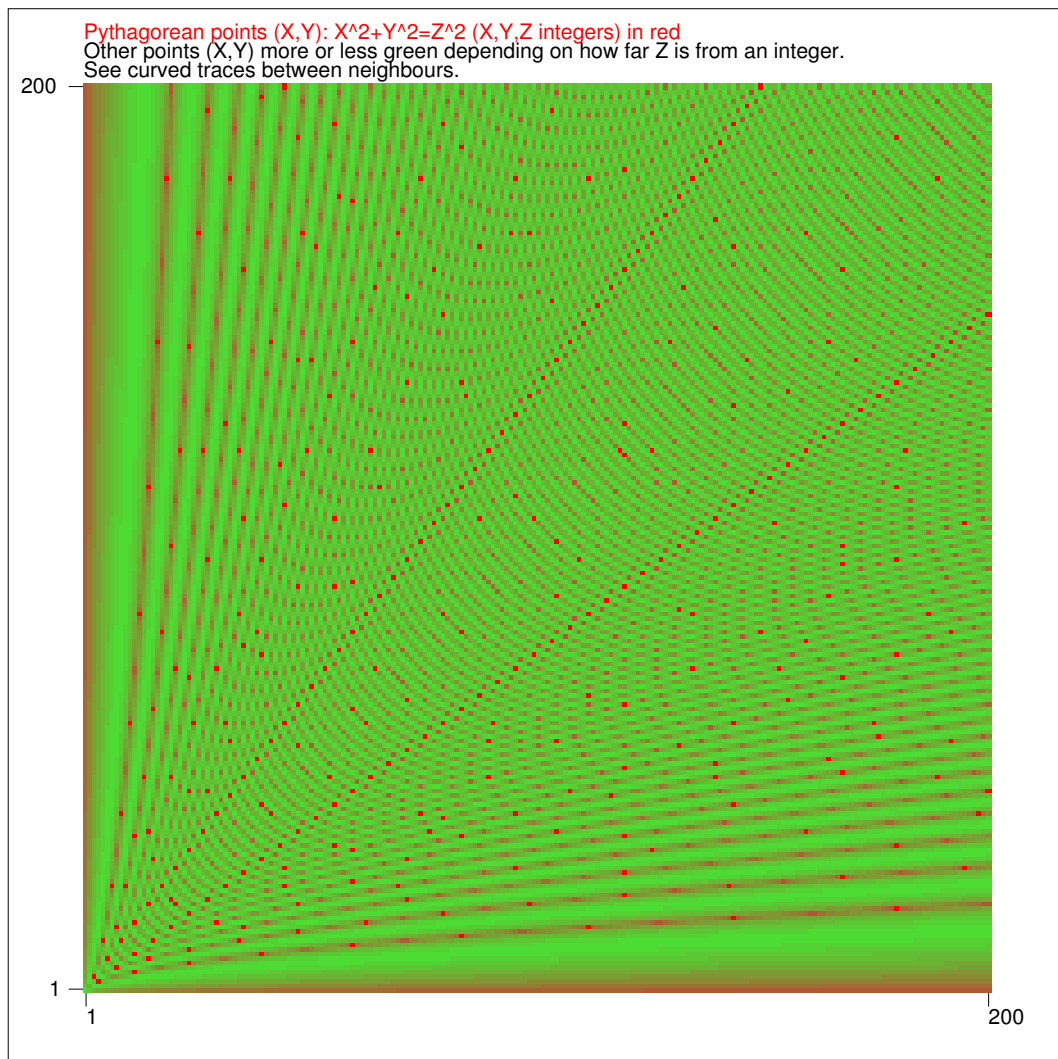


PYTHAGOREAN TRIPLES: VISUALIZATION AND CHARACTERIZATION

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Our starting points are “tiny red flowers on a green meadow”:



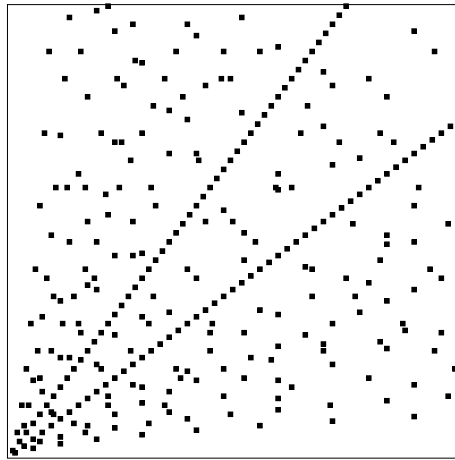
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Recently I extended graphical capabilities of the Survo system [4] by a feature that makes possible to define the color of a point in a scatter diagram of variables, say \mathbf{X} and \mathbf{Y} , by integer values of any variable, say \mathbf{F} , in the current data set.

As one illustration of this enhancement I decided to make such a plot of variables \mathbf{X}, \mathbf{Y} having integer values $1, 2, \dots, 200$ and color a point (\mathbf{X}, \mathbf{Y}) into 'red' if \mathbf{X} and \mathbf{Y} belong to a Pythagorean triple of integers $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ where $\mathbf{Z}^2 = \mathbf{X}^2 + \mathbf{Y}^2$. Other points would be colored on a scale from 'red' to 'green' depending on how much $\mathbf{Z} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2}$ differs from the nearest integer on a nonlinear scale on the interval $(0, 0.5)$ (see details in the appendix).

The picture on the first page tells more than displays on the same topic of the following style shown e.g. in [2].



Since in the colored graph 'almost Pythagorean' points become more or less 'red-dish', interesting traces connecting true Pythagorean points will be seen.

This illustration revealed to me e.g. that all Pythagorean points (\mathbf{X}, \mathbf{Y}) seem to be located on hyperbolas

$$\mathbf{Y} = 2\mathbf{C}(\mathbf{X} - \mathbf{C})/(\mathbf{X} - 2\mathbf{C}), \quad \mathbf{C} = 1, 2, \dots$$

To get them plotted, their equations can be written in parametric form

$$(1) \quad \mathbf{X}(\mathbf{t}) = \mathbf{C} \sqrt{2/\tan(\mathbf{t})} + 2\mathbf{C}, \quad \mathbf{Y}(\mathbf{t}) = \mathbf{C} \sqrt{2 \tan(\mathbf{t})} + 2\mathbf{C}, \quad 0 < \mathbf{t} < \pi/2$$

and now these curves will be seen (in yellow) for values $\mathbf{C}=1,2,\dots,50$ in the next graph.

Obviously the number $\mathbf{N}(\mathbf{C})$ of Pythagorean points on the curve

$$\mathbf{Y} = 2\mathbf{C}(\mathbf{X} - \mathbf{C})/(\mathbf{X} - 2\mathbf{C})$$

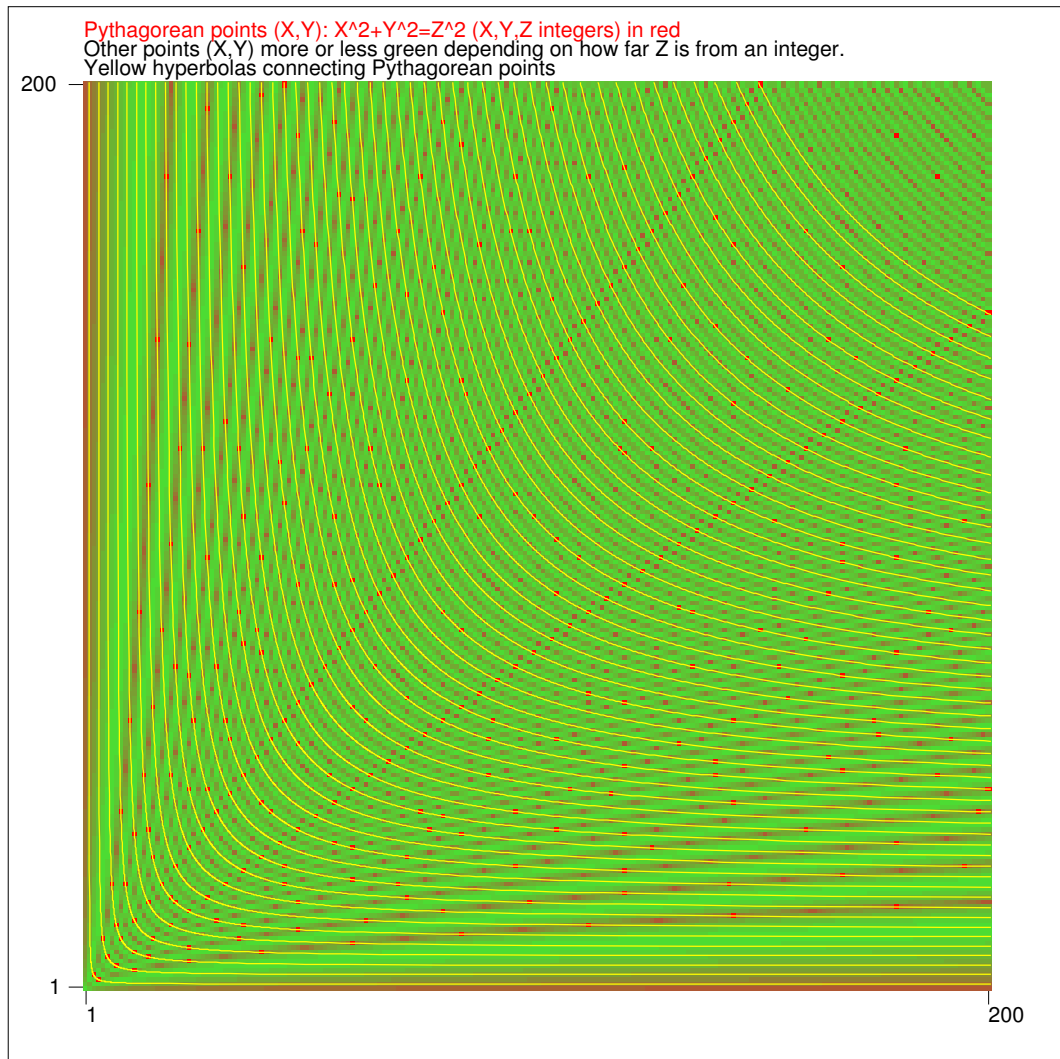
is finite and by direct computation we get for $\mathbf{C} = 1, 2, \dots, 16$

\mathbf{C}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\mathbf{N}(\mathbf{C})$	2	4	6	6	6	12	6	8	10	12	6	18	6	12	18	10

Numbers $\mathbf{N}(\mathbf{C})/2$ seem to form the sequence A078644 in Sloane's Encyclopedia of Integer Sequences [3] and thus the general expression for $\mathbf{N}(\mathbf{C})$ is

$$\mathbf{N}(\mathbf{C}) = \tau_0(2\mathbf{C}^2)$$

where $\tau_0(\mathbf{n})$ is the number of divisors of \mathbf{n} .



We prove this statement in the following way:

Since $\mathbf{X}(\mathbf{t})$ and $\mathbf{Y}(\mathbf{t})$ must be positive integers, expressions $\sqrt{2/\tan(\mathbf{t})}$ and $\sqrt{2\tan(\mathbf{t})}$ must be rational and this implies that $\tan(\mathbf{t}) = 2\mathbf{k}^2$ where \mathbf{k} is a rational number.

Then equations (1) can be written in the form

$$(2) \quad \mathbf{X}(\mathbf{k}) = 2\mathbf{C} + \mathbf{C}/\mathbf{k},$$

$$(3) \quad \mathbf{Y}(\mathbf{k}) = 2\mathbf{C} + 2\mathbf{C}\mathbf{k}.$$

The last member of a Pythagorean triple $\mathbf{Z}(\mathbf{k})$ ($\mathbf{Z}(\mathbf{k})^2 = \mathbf{X}(\mathbf{k})^2 + \mathbf{Y}(\mathbf{k})^2$) is

$$(4) \quad \mathbf{Z}(\mathbf{k}) = 2\mathbf{C}\mathbf{k} + 2\mathbf{C} + \mathbf{C}/\mathbf{k} = \mathbf{X}(\mathbf{k}) + \mathbf{Y}(\mathbf{k}) - 2\mathbf{C}.$$

Then $\mathbf{N}(\mathbf{C})$ will be the number of \mathbf{k} -values for which all three expressions (2), (3), (4) are positive integers.

Since, according to (4), $\mathbf{Z}(\mathbf{k})$ is an integer when $\mathbf{X}(\mathbf{k})$ and $\mathbf{Y}(\mathbf{k})$ are integers, it is sufficient to pay attention to $\mathbf{X}(\mathbf{k})$ and $\mathbf{Y}(\mathbf{k})$ only.

If \mathbf{C} is a prime number $\mathbf{p} \neq 2$, it is seen immediately that the only accepted values of \mathbf{k} , giving integer values for $\mathbf{X}(\mathbf{k}), \mathbf{Y}(\mathbf{k})$ i.e. simultaneously for \mathbf{C}/\mathbf{k} and $2\mathbf{C}\mathbf{k}$, are

$$\mathbf{k} = 1, \mathbf{p}, 1/\mathbf{p}$$

and their halves

$$\mathbf{k} = 1/2, \mathbf{p}/2, 1/2\mathbf{p}$$

since the latter ones give an integer value also to $2\mathbf{C}\mathbf{k}$.

Similarly, if \mathbf{C} is product of two primes $\mathbf{p}, \mathbf{q} \neq 2$, the only valid values for \mathbf{k} are

$$\mathbf{k} = 1, \mathbf{p}, \mathbf{q}, \mathbf{pq}, \mathbf{p}/\mathbf{q}, \mathbf{q}/\mathbf{p}, 1/\mathbf{p}, 1/\mathbf{q}, 1/\mathbf{pq}$$

and their halves

$$\mathbf{k} = 1/2, \mathbf{p}/2, \mathbf{q}/2, \mathbf{pq}/2, \mathbf{p}/2\mathbf{q}, \mathbf{q}/2\mathbf{p}, 1/2\mathbf{p}, 1/2\mathbf{q}, 1/2\mathbf{pq}.$$

Now it is crucial to notice that there is a bijection from divisors of $2\mathbf{C}^2 = 2\mathbf{p}^2\mathbf{q}^2$ to these values of \mathbf{k} by

$$2^i \mathbf{p}^{i_p} \mathbf{q}^{i_q} \Leftrightarrow 2^{i-1} \mathbf{p}^{i_p-1} \mathbf{q}^{i_q-1}$$

where $\mathbf{i} = 0, 1, \mathbf{i}_p, \mathbf{i}_q = 0, 1, 2$.

In the general case let the prime factorization of \mathbf{C} be $\mathbf{C} = 2^a \mathbf{p}_1^{a_1} \dots \mathbf{p}_m^{a_m}$. Then the bijection is

$$(5) \quad 2^i \mathbf{p}_1^{i_1} \dots \mathbf{p}_m^{i_m} \Leftrightarrow 2^{i-a-1} \mathbf{p}_1^{i_1-a_1} \dots \mathbf{p}_m^{i_m-a_m}$$

where $\mathbf{i} = 0, 1, \dots, 2\mathbf{a} + 1, \mathbf{i}_1 = 0, 1, \dots, 2\mathbf{a}_1, \dots, \mathbf{i}_m = 0, 1, \dots, 2\mathbf{a}_m$.

Since the only valid \mathbf{k} -values (making $\mathbf{X}(\mathbf{k})$ and $\mathbf{Y}(\mathbf{k})$ integers) are values on the right-hand side of (5) and the left-hand side gives all divisors of $2\mathbf{C}^2$, we finally obtain

$$\mathbf{N}(\mathbf{C}) = \mathbf{d}(2\mathbf{C}^2).$$

The same result has been recently derived by McCullough [1] but our treatment of the problem is simpler due to the parametric representation (2),(3) while McCullough is operating in a more general framework. Also the nice interpretation of $\mathbf{N}(\mathbf{C})$ as the number of divisors of $2\mathbf{C}^2$ seems not to be mentioned in his paper.

From (4) we get the excess $\mathbf{X}(\mathbf{T}) + \mathbf{Y}(\mathbf{T}) - \mathbf{Z}(\mathbf{T}) = 2\mathbf{C}$ and this is the basis of enumeration in McCullough's paper. Since the excess is always an even integer, it is clear that only integer values of \mathbf{C} give Pythagorean triples. Hence equations (2), (3), (4) give an exhaustive enumeration of these triples for $\mathbf{C} = 1, 2, \dots$ and for various \mathbf{k} -values obtained from divisors of $2\mathbf{C}^2$ according to (5).

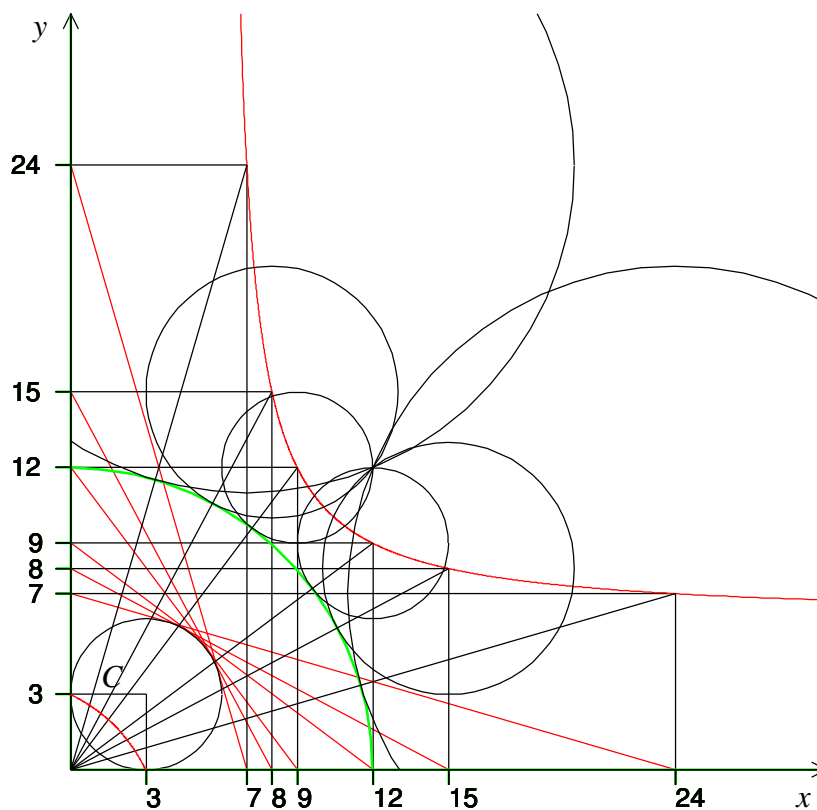
Geometric interpretation (17 June 2005)

As a personal communication Darryl McCullough wrote me (11 June 2005):
 “ C is the radius of the incircle, so place the incircle in the first quadrant tangent to the x - and y -axes at $x = C$ and $y = C$. Then, the hyperbola is the locus of points (X, Y) such that X and Y are the intercepts of a line tangent to the incircle.”

Thus $N(C)$ is also the number of Pythagorean triples with a given radius C of the incircle.

It can be concluded further that the hyperbola has its loci in points $(0, 0)$ and $(4C, 4C)$ and the difference of distances of any point on the hyperbola from these loci is $4C$. Then the circles $X^2 + Y^2 = (4C)^2$ and $(X - X_0)^2 + (Y - Y_0)^2 = R^2$, where (X_0, Y_0) are the legs of a right triangle with the radius C of the incircle and $R^2 = (X_0 - 4C)^2 + (Y_0 - 4C)^2$, are mutually tangent and the line from $(0, 0)$ to (X_0, Y_0) goes through the tangent point of those circles.

The following graph illustrates the case $C = 3$ giving six Pythagorean triples $(7, 24, 25)$, $(8, 15, 17)$, $(9, 12, 25)$, $(12, 9, 15)$, $(15, 8, 17)$, $(24, 7, 25)$.



Appendix:**Making the graphics and calculations as Survo applications**

As I mentioned earlier, this study is a consequence of a small enhancement made in the Survo system [4]. It is worthwhile to show how Survo has given support for making the graphics and calculations needed in this presentation. In fact, the entire manuscript has been prepared with the Survo editor into \LaTeX format.

The grid of points $\{(\mathbf{X}, \mathbf{Y}) \mid \mathbf{X}, \mathbf{Y} = 1, 2, \dots, 200\}$ is created as a Survo data file TRIPLES and variables \mathbf{Z} , \mathbf{D} (deviation of \mathbf{Z} from the nearest integer), and \mathbf{F} (the color index) are computed in the Survo edit field as follows:

```
-----
Creating a data file TRIPLES:
FILE CREATE TRIPLES,64,8
FIELDS:
1 N 2 X
2 N 2 Y
3 N 4 Z
END

FILE INIT TRIPLES,40000 / Space for 200*200=40000 observations

Creating a 200*200 grid of points (X,Y):
M=200
X=int((ORDER-1)/M)+1
Y=ORDER-X*M+M
VAR X,Y TO TRIPLES / Computing X and Y values

Computing Z values:
VAR Z=sqrt(X*X+Y*Y) TO TRIPLES

Computing deviation D from the nearest integer:
D=min(Z-int(Z),int(Z+1)-Z)
VAR D TO TRIPLES

Non-linear mapping of D values to color indices -21,-20,...,-1:
F=-int(21*(2*D)^0.2)-1
VAR F:2 TO TRIPLES
-----
```

The picture on the first page is plotted as a PostScript file PYTH1.PS by the following Survo PLOT scheme:

```
-----
PLOT TRIPLES,X,Y / DEVICE=PS,PYTH1.PS
POINT=6,4 POINT_COLOR=F (This is the "main point"!)

SCALE=0.5:?,1,200,200.5:? XLABEL= YLABEL= HEADER=
SIZE=1270,1270
PEN=[Swiss(7.26)]
XDIV=100,1200,100 YDIV=100,1200,100
TEXTS=T1,T2,T3
T1=[RED],Pythagorean_points_(X;Y):_X^2+Y^2=Z^2_&
(X;Y;Z_integers)_in_red,91,1234
T2=[BLACK],Other_points_(X;Y)_more_or_less_green_&
depending_on_how_far_Z_is_from_an_integer.,91,1211
T3=[BLACK],See_curved_traces_between_neighbours.,91,1188
```

```
21 colors gradually from red to green:(cmyk codes)
FILL(-1)=0.00,1.00,1.0,0 FILL(-11)=0.40,0.54,0.8,0
FILL(-2)=0.13,0.90,0.8,0 FILL(-12)=0.43,0.50,0.8,0
FILL(-3)=0.16,0.86,0.8,0 FILL(-13)=0.46,0.46,0.8,0
FILL(-4)=0.19,0.82,0.8,0 FILL(-14)=0.49,0.42,0.8,0
FILL(-5)=0.22,0.78,0.8,0 FILL(-15)=0.52,0.38,0.8,0
FILL(-6)=0.25,0.74,0.8,0 FILL(-16)=0.55,0.34,0.8,0
FILL(-7)=0.28,0.70,0.8,0 FILL(-17)=0.58,0.30,0.8,0
FILL(-8)=0.31,0.66,0.8,0 FILL(-18)=0.61,0.26,0.8,0
FILL(-9)=0.34,0.62,0.8,0 FILL(-19)=0.64,0.22,0.8,0
FILL(-10)=0.37,0.58,0.8,0 FILL(-20)=0.67,0.18,0.8,0
FILL(-21)=0.70,0.14,0.8,0
```

```
-----
The hyperbolas (1) are plotted in yellow for  $C = 1, 2, \dots, 50$  as a PostScript
file PYTH2.PS and combined with PYTH1B.PS, a modified version (partially different
texts) of the first figure:
```

```
-----
Plotting the hyperbolas:
PLOT X(t)=C*sqrt(2/tan(t))+2*C-1/4,Y(t)=C*sqrt(2*tan(t))+2*C-1/4
DEVICE=PS,PYTH2.PS
```

```
C=1(1)50
pi=3.141592653589793
T=[YELLOW][line_width(0.48)],0.00001,pi/2,pi/1000
```

```
FRAME=0
SCALE=0.5,200.5 XLABEL= YLABEL= HEADER=
SIZE=1270,1270
XDIV=100,1200,100 YDIV=100,1200,100
```

```
Making a combined graph PYTH12.PS:
EPS JOIN PYTH12.PS PYTH1B.PS PYTH2.PS
-----
```

For finding $N(C)$ -values (before conceiving the general rule by consulting Sloane's Encyclopedia) I made a Survo macro (sucro) /NPYTH. For $C = 864$ it gives the following list of Pythagorean triples (X, Y, Z) for $X < Y$:

```
-----
/NPYTH 864
Pythagorean triples for C=864: N(C)/2=42 triples found!
(2880,3024,4176) (2752,3186,4210) (2700,3264,4236) (2592,3456,4320)
(2496,3672,4440) (2457,3776,4505) (2376,4032,4680) (2304,4320,4896)
(2240,4644,5156) (2214,4800,5286) (2160,5184,5616) (2112,5616,6000)
(2052,6336,6660) (2016,6912,7200) (1984,7560,7816) (1971,7872,8115)
(1944,8640,8856) (1920,9504,9696) (1890,10944,11106) (1872,12096,12240)
(1856,13392,13520) (1836,15552,15660) (1824,17280,17376)
(1809,20160,20241) (1800,22464,22536) (1792,25056,25120)
(1782,29376,29430) (1776,32832,32880) (1764,43200,43236)
(1760,48384,48416) (1755,57024,57051) (1752,63936,63960)
(1746,84672,84690) (1744,95040,95056) (1740,126144,126156)
(1737,167616,167625) (1736,188352,188360) (1734,250560,250566)
(1732,374976,374980) (1731,499392,499395) (1730,748224,748226)
(1729,1494720,1494721)
```

Checking the result by editorial computing of Survo:

$2 \cdot C^2 = 1492992$

$1492992(10:\text{factors}) = 2^7 \cdot 11 \cdot 3^6 \quad (11+1) \cdot (6+1) / 2 = 42$

Here is the sucro code listing:

```
-----
*TUTSAVE NPYTH
/ /NPYTH C
/ lists all Pythagorean triples (X<Y,Z) located on the hyperbola
/ Y = 2C(X-C)/(X-2C)
/ for any integer C.
/
/ def Wc=W1 Wx=W2 Wy=W3 Wz=W4 Wn=W5 Wt=W6
/
*{tempo -1}{R}{erase}{ref set 1}{R}{erase}{ref set 2}
/ Starting from maximum value of X+1:
*int((2+sqrt(2))*{print Wc})+1={act}{1} {save word Wx}{line start}{erase}
*{Wn=0}
/
+ A: {ref jump 1}{line start}{erase}{Wx=Wx-1}
- if Wx < 1 then goto E
*{print Wx}-2*{print Wc}={act}{1} {save word Wt}
- if Wt <= 0 then goto E
*2*{print Wc}*({print Wx}-{print Wc})/({print Wx}-2*{print Wc})={act}
*{1} {find .}{save char Wt}
/ If Wy is not integer, try a new X:
- if Wt '=' . then goto A
*{save word Wy}{line start}{erase}
```



```

*sqrt({print Wx}^2+{print Wy}^2)={act}{1} {find .}{save char Wt}
/ If Wz is not integer, try a new X:
- if Wt '=' . then goto A
/ A new triple is found. Add it to the list:
*{save word Wz}
* {line start}{erase}({print Wx},{print Wy},{print Wz}){line start}
*{save line Wt}
*{ref jump 2}{Wn=Wn+1}
*{write Wt}
* {ref set 2}
*{ref jump 1}{goto A}
/ Final remarks:
+ E: {ref jump 1}{erase}Pythagorean triples for C={print Wc}:
* N(C)/2={print Wn} triples found!
*{end}

```

I have also made a more extensive `sucro P_TRIPLE` which plots the “tiny red flowers” on the screen. The user can pick a “flower” by the mouse and the `sucro` tells whether it corresponds to a Pythagorean triple. In any case it displays the **X**, **Y**, and **Z** values. The procedure may be repeated. `P_TRIPLE` is available in `SURVO MM` (ver. 2.28+).

The current version of this paper can be downloaded from
<http://www.survo.fi/papers/pythagorean3.pdf>

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