Some polynomials associated with regular polygons

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Abstract. Let \( G_n \) be a regular \( n \)-gon with unit circumradius, and \( m = \lfloor \frac{n}{2} \rfloor \), \( \mu = \lfloor \frac{n-1}{2} \rfloor \). Let the edges and diagonals of \( G_n \) be \( e_{n1} < \cdots < e_{nm} \). We compute the coefficients of the polynomial
\[
(x - e_{n1}^2) \cdots (x - e_{n\mu}^2).
\]
They appear to form a well-known integer sequence, and we study certain related sequences, too. We also compute the coefficients of the polynomial
\[
(x - s_{n1}^2) \cdots (x - s_{nm}^2),
\]
where
\[
s_{ni} = \cot \left( \frac{2i - 1}{2n} \pi \right), \quad i = 1, \ldots, m.
\]
We interpret \( s_{n1} \) as the sum of all individual edges and diagonals of \( G_n \) divided by \( n \). We also discuss the interpretation of \( s_{n2}, \ldots, s_{nm} \), and present a conjecture on expressing \( s_{n1}, \ldots, s_{nm} \) using \( e_{n1}, \ldots, e_{nm} \).

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1 Introduction

Throughout, $G_n$ is a regular $n$-gon with unit circumradius, and

$$m = \left\lfloor \frac{n}{2} \right\rfloor, \quad \mu = \left\lfloor \frac{n-1}{2} \right\rfloor.$$  

Long time ago Kepler observed [2] that the squares of the edge and diagonals of $G_7$ are the zeros of the polynomial $x^3 - 7x^2 + 14x - 7$. This raises a general question: Are the squares of (the lengths of) the edge and diagonals of $G_n$, excluding the diameter, the zeros of a monic polynomial of degree $\mu$ with integer coefficients?

Yes, they are. This follows from Savio’s and Suruyanarayan’s [6] results, which, however, do not give the polynomial explicitly. We will do it in Section 2. A natural further question concerns the edge and diagonals themselves, instead of their squares. They are not zeros of a polynomial described above, but we will in Section 3 see that the squared sum of all individual edges and diagonals is the largest zero of a monic polynomial of degree $m$ with integer coefficients. We will study geometric interpretation of the square roots of the other zeros in Section 4. In Section 5, we will present a conjecture on expressing these square roots as simple linear combinations of the edge and diagonals. We will in Section 6 notify that the coefficients of the first-mentioned polynomial form an OEIS [4] sequence, and also study OEIS sequences corresponding to certain related polynomials. Finally, we will complete our paper with conclusions and further questions in Section 7.

2 Squared chords

Let (the lengths of) the edge and diagonals of $G_n$ be $e_{n1} < \cdots < e_{nm}$. Call them (the lengths of) the chords. Then

$$e_{nk} = 2 \sin \frac{k\pi}{n}, \quad k = 1, \ldots, m.$$  

Our problem is to find the coefficients $a_{mk}$ and $b_{mk}$ of the polynomials

$$A_m(x) = (x - e_{n+2,1}^2) \cdots (x - e_{n+2,m}^2) = x^m + a_{m,m-1}x^{m-1} + \cdots + a_{m1}x + a_{m0}, \quad (1)$$

where $n$ is even, and

$$B_m(x) = (x - e_{n1}^2) \cdots (x - e_{nm}^2) = x^m + b_{m,m-1}x^{m-1} + \cdots + b_{m1}x + b_{m0}, \quad (2)$$
where $n$ is odd. We solve it in two theorems. Mustonen [3] found them experimentally and sketched their proofs.

Let $\text{tridiag}_m(x, y)$ denote the symmetric tridiagonal $m \times m$ matrix with all main diagonal entries $x$ and first super- and subdiagonal entries $y$. For $m \geq 2$, define

$$A_m = \text{tridiag}_m(2, 1)$$

and

$$B_m$$

is as $A_m$ but the $(m, m)$ entry equals 3.

Also define $A_1 = (2)$ and $B_1 = (3)$. Denote by spec the (multi)set of eigenvalues.

**Lemma 1** For all $m \geq 1$,

$$\text{spec } A_m = \left\{ 4 \sin^2 \frac{k\pi}{n+2} \mid k = 1, \ldots, m \right\} = \{ e_{n+2,1}^2, \ldots, e_{n+2,m}^2 \},$$

$$\text{spec } B_m = \left\{ 4 \sin^2 \frac{k\pi}{n} \mid k = 1, \ldots, m \right\} = \{ e_{n1}^2, \ldots, e_{nm}^2 \}.$$  

**Proof.** See [1, 5, 6].

**Theorem 1** In (1),

$$a_{mk} = (-1)^{m-k} \binom{m + 1 + k}{2k + 1}.$$  

**Proof.** Denoting

$$p_m(x) = x^m + \sum_{k=0}^{m-1} (-1)^{m-k} \binom{m + 1 + k}{2k + 1} x^k,$$

our claim is that

$$p_m(x) = A_m(x)$$

for all $m \geq 1$. Expanding $\det (xI_m - A_m)$ along the last row, we have

$$A_{m+1}(x) = (x - 2)A_m(x) - A_{m-1}(x)$$

for all $m \geq 2$. Since

$$p_1(x) = x - 2 = A_1(x)$$
and

\[ P_2(x) = x^2 - 4x + 3 = A_2(x), \]

the claim (5) follows by showing that

\[ P_{m+1}(x) = (x - 2)P_m(x) - P_{m-1}(x) \tag{6} \]

for all \( m \geq 2 \). Mustonen [3] did it by using Mathematica. We will do the computations algebraically in the appendix. □

The formula (4) yields \( a_{mm} = 1 \), consistently with the coefficient of \( x^m \) in (1). It also allows to define \( a_{00} = 1 \). The polynomial

\[ \tilde{A}_{m+1}(x) = (x - 4)A_m(x) = x^{m+1} + \alpha_{m+1,m}x^m + \cdots + \alpha_{m+1,1}x + \alpha_{m+1,0} \tag{7} \]

has \( e_{n+2,m+1}^2 = 4 \) as the additional zero. By (4),

\[ \alpha_{m+1,k} = (-1)^{m-k+1} \left( \binom{m+k}{2k-1} + 4 \binom{m+1+k}{2k+1} \right). \tag{8} \]

(We define \( \binom{n}{k} = 0 \) if \( k < 0 \).)

**Theorem 2**  In (2),

\[ b_{mk} = (-1)^{m-k} \frac{2m+1}{m-k} \frac{m+k}{2k+1} = (-1)^{m-k} \left( \binom{m+1+k}{2k+1} + \binom{m+k}{2k+1} \right). \tag{9} \]

**Proof.** The second equation follows from trivial computation. To show the first, denote

\[ Q_m(x) = x^m + \sum_{k=0}^{m-1} (-1)^{m-k} \frac{2m+1}{m-k} \frac{m+k}{2k+1} x^k \]

and claim that

\[ Q_m(x) = B_m(x) \tag{10} \]

for all \( m \geq 1 \). Expanding \( \det (xI_m - B_m) \), we have

\[ B_{m+1}(x) = (x - 3)A_m(x) - A_{m-1}(x) \]

for all \( m \geq 2 \). Since

\[ Q_1(x) = x - 3 = B_1(x) \]
and

\[ Q_2(x) = x^2 - 5x + 5 = B_2(x), \]

the claim (10) follows by showing that

\[ Q_{m+1}(x) = (x - 3)P_m(x) - P_{m-1}(x) \tag{11} \]

for all \( m \geq 2 \). Mustonen [3] did also this by using Mathematica, and we will do the computations algebraically in the appendix. □

For \( k = m \), the first expression in (9) is undefined but the second is defined. (We define \( \binom{n}{k} = 0 \) if \( n < k \).) It gives \( b_{mm} = 1 \), the coefficient of \( x^m \) in (2). It also allows to define \( b_{00} = 1 \).

**Corollary 1** The sum of all individual squared chords of \( G_n \) is \( n^2 \). Their product is \( n^n \).

**Proof.** By Theorems 1 and 2 (or by [7, Eqs. (20) and (24)]), we obtain

\[
\begin{align*}
& e_{2m,1}^2 + \cdots + e_{2m,m-1}^2 = -a_{m-1,m-2} = 2(m-1), \\
& e_{2m+1,1}^2 + \cdots + e_{2m+1,m}^2 = -b_{m,m-1} = 2m + 1,
\end{align*}
\]

and

\[
\begin{align*}
& e_{2m,1}^2 \cdots e_{2m,m-1}^2 = (-1)^m a_{m-1,0} = m, \\
& e_{2m+1,1}^2 \cdots e_{2m+1,m}^2 = (-1)^m b_{m0} = 2m + 1.
\end{align*}
\]

Denoting by \( \Sigma_n \) the sum and by \( \Pi_n \) the product of all individual squared chords of \( G_n \), we therefore have

\[
\begin{align*}
\Sigma_{2m} &= 2m \cdot 2(m - 1) + m \cdot 4 = (2m)^2, \\
\Sigma_{2m+1} &= (2m + 1)(2m + 1) = (2m + 1)^2,
\end{align*}
\]

and

\[
\begin{align*}
\Pi_{2m} &= m^2 m^4 = (2m)^2m, \\
\Pi_{2m+1} &= (2m + 1)^{2m+1}.
\end{align*}
\]

□
3 Sum of chords

The sum of all individual chords of $G_n$ is

$$S_n = n s_n,$$

where

$$s_n = e_{n1} + \cdots + e_{n,m-1} + \frac{1}{2} e_{nm} = e_{n1} + \cdots + e_{n,m-1} + 1$$

if $n$ is even, and

$$s_n = e_{n1} + \cdots + e_{nm}$$

if $n$ is odd, is the sum of different (lengths of) chords but the diameter is halved.

**Theorem 3** For all $n \geq 3$,

$$s_n = \cot \frac{\pi}{2n}.$$  

**Proof.** We have [7, Eq. (21)]

$$\sum_{k=1}^{n-1} \sin \frac{k\pi}{n} = \cot \frac{\pi}{2n}. \quad (12)$$

If $n$ is even, this implies

$$s_n = \sum_{k=1}^{m-1} 2 \sin \frac{k\pi}{n} + \frac{1}{2} 2 = \sum_{k=1}^{m-1} \sin \frac{k\pi}{n} + 1 + \sum_{k=m+1}^{2m-1} \sin \frac{k\pi}{n} =$$

$$\sum_{k=1}^{2m-1} \sin \frac{k\pi}{n} = \cot \frac{\pi}{2n}.$$  

If $n$ is odd, then

$$s_n = \sum_{k=1}^{m} 2 \sin \frac{k\pi}{n} = \sum_{k=1}^{m} \sin \frac{k\pi}{n} + \sum_{k=m+1}^{2m} \sin \frac{k\pi}{n} = \sum_{k=1}^{2m} \sin \frac{k\pi}{n} = \cot \frac{\pi}{2n}.$$  

\[\square\]

Is $s_n$ a zero of a monic polynomial of degree $m$ with integer coefficients? Yes for $s_4 = \cot \frac{\pi}{8} = 1 + \sqrt{2}$; it is a zero of $x^2 - 2x - 1$. On the other hand, it is easy to see that $s_5 = \cot \frac{\pi}{10} = \sqrt{5} + 2\sqrt{5}$ is not a zero of such a polynomial. But
$s_2^2 = 5 + 2\sqrt{5}$ is a zero of $x^2 - 10x + 5$, and the other zero is $5 - 2\sqrt{5} = \cot^2 \frac{3\pi}{10}$.

Also $s_4^2 = 3 + 2\sqrt{2}$ has this property: it is a zero of $x^2 - 6x + 1$, and the other zero is $3 - 2\sqrt{2} = \cot^2 \frac{3\pi}{8}$.

Generally, denoting $s_n = \cot \left( \frac{(2i - 1)\pi}{2n} \right)$, $i = 1, \ldots, m$,

this motivates us to study for even $n$ the coefficients of the polynomial

$$U_m(x) = (x - s_{n1}^2) \cdots (x - s_{nm}^2) = x^m + u_{m,m-1}x^{m-1} + \cdots + u_{m1}x + u_{m0}, \quad (13)$$

and for odd $n$ those of

$$V_m(x) = (x - s_{n1}^2) \cdots (x - s_{nm}^2) = x^m + v_{m,m-1}x^{m-1} + \cdots + v_{m1}x + v_{m0}. \quad (14)$$

We will see that they all are integers. The largest zero is $s_n^2 = s_{n1}^2$.

Mustonen [3] found the following theorem experimentally and also presented its proof. Yaglom and Yaglom [9, Eqs. (7) and (8)] formulated (16) differently.

**Theorem 4** In (13),

$$u_{mk} = (-1)^k \binom{n}{2k}. \quad (15)$$

In (14),

$$v_{mk} = (-1)^k \binom{n}{2k+1}. \quad (16)$$

**Proof.** We have [10]

$$\cot nt = \frac{\sum_{k=0}^{m} (-1)^k \binom{n}{2k} \cot^{n-2k} t}{\sum_{k=0}^{m} (-1)^k \binom{n}{2k+1} \cot^{n-2k-1} t}. \quad (17)$$

Denote

$$t_i = \frac{(2i - 1)\pi}{2n}, \quad i = 1, \ldots, m.$$

Since $\cot nt_i = 0$, (17) yields

$$\sum_{k=0}^{m} (-1)^k \binom{n}{2k} \cot^{n-2k} t_i = 0. \quad (18)$$
First assume $n$ even. The polynomial
\[
\tilde{U}_m(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{n}{2k} x^k
\]
is monic and has degree $m$. For all $i = 1, \ldots, m$,
\[
\tilde{U}_m(s_{ni}^2) = \sum_{k=0}^{m} (-1)^{m-k} \binom{2m}{2k} s_{ni}^{2k} = \sum_{l=0}^{m} (-1)^{l} \binom{2m}{2m-2l} s_{ni}^{2m-2l}
\]
by (18). Hence
\[
\tilde{U}_m(x) = (x - s_{n1}^2) \cdots (x - s_{nm}^2) = U_m(x),
\]
and (15) follows.

Second, assume $n$ odd. The polynomial
\[
\tilde{V}_m(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{n}{2k+1} x^k
\]
is monic and has degree $m$. For all $i = 1, \ldots, m$,
\[
\tilde{V}_m(s_{ni}^2) = \sum_{k=0}^{m} (-1)^{m-k} \binom{2m+1}{2k+1} s_{ni}^{2k} = \sum_{l=0}^{m} (-1)^{l} \binom{2m+1}{2m-2l+1} s_{ni}^{2m-2l}
\]
\[
= s_{ni}^{-1} \sum_{l=0}^{m} (-1)^{l} \binom{n}{2l} \cot^{n-2l} t_i = 0,
\]
again by (18). Hence
\[
\tilde{V}_m(x) = (x - s_{n1}^2) \cdots (x - s_{nm}^2) = V_m(x),
\]
and (16) follows. \qed
Corollary 2 The number $s^2_n$ is the largest zero of the polynomial

$$x^m + u_{m,m-1}nx^{m-1} + \cdots + u_{m1}n^{m-1}x + u_0n^m$$

if $n$ is even, and that of

$$x^m + v_{m,m-1}nx^{m-1} + \cdots + v_{m1}n^{m-1}x + v_0n^m$$

if $n$ is odd.

4 Interpreting $s_{n,m-k+1}$, $k = 1, \ldots, \lfloor \frac{n-1}{3} \rfloor$, $n$ odd

The zeros of $A_m(x)$ and $B_m(x)$ describe the squared chords of $G_{2m+2}$ and $G_{2m+1}$, respectively, excluding the diameter. The largest zero of $U_m(x)$, $s^2_{2m,1} = s^2_{2m}$, and that of $V_m(x)$, $s^2_{2m+1,1} = s^2_{2m+1}$, describe the squared sum of chords but halving the diameter. In other words, the sum of all individual chords of $G_n$ is divided by $n$ and the result is squared.

What about the other zeros?

Let the vertices of $G_n$ be $P_0, \ldots, P_{n-1}$, where $P_k = (\cos \frac{k\pi}{n}, \sin \frac{k\pi}{n})$. Then $e_{nk} = P_0P_k = 2\sin \frac{k\pi}{n}$, $k = 1, \ldots, m$. Since $P_0P_{n-k} = P_0P_k$, we define $e_{n,n-k} = e_{nk}, k = 1, \ldots, m$.

Fix $n$ and denote $e_k = e_{nk}$ for brevity. Assume that $3k < n$; i.e., $k < \frac{n}{3}$. Then the line segments $P_0P_{2k}$ and $P_kP_{n-k}$ intersect; let $Q_k$ be their intersection point and denote $x_k = P_0Q_k$. Because $\triangle Q_kP_0P_k \sim \triangle Q_kP_{2k}P_{n-k}$, we have

$$\frac{x_k}{e_{2k} - x_k} = \frac{e_k}{e_{3k}}.$$

Hence

$$x_k = \frac{e_k e_{2k}}{e_k + e_{3k}} = \frac{2\sin \frac{k\pi}{n} \sin \frac{2k\pi}{n}}{\sin \frac{k\pi}{n} + \sin \frac{3k\pi}{n}} = \frac{\sin \frac{k\pi}{n} \sin \frac{2k\pi}{n}}{\sin \frac{2k\pi}{n} \cos \frac{k\pi}{n}} = \tan \frac{k\pi}{n}.$$

If $n$ is odd, then

$$\tan \frac{k\pi}{n} = \cot \left( \frac{\pi}{2} - \frac{k\pi}{2m+1} \right) = \cot \left[ \frac{2(m-k) + 1}{2n} \right] = s_{n,m-k+1}.$$
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Thus \( s_{n,m-k+1} = P_0 Q_k, \ k = 1, \ldots, \lfloor \frac{n-1}{3} \rfloor \). In other words, the \( \lfloor \frac{n-1}{3} \rfloor \) smallest zeros of \( V_m(x) \) are the squared line segments \( P_0 Q_k, \ k = 1, \ldots, \lfloor \frac{n-1}{3} \rfloor \). Mustonen [3] found this experimentally. The largest zero is already interpreted, but the interpretation of the rest of zeros remains open. For some experimental observations, see [3]. Interpretation of the zeros of \( U_m(x) \), except the largest, remains open, too.

5 Expressing \( s_{n1}, \ldots, s_{nm} \) using \( e_{n1}, \ldots, e_{nm} \)

Mustonen’s [3] experiments make conjecture that, given \( n \), there are numbers \( \lambda_{nk}^{(i)} \in \{0, \pm 1\}, \ i, k = 1, \ldots, m \), such that

\[
\begin{align*}
\text{sn}_i &= \lambda_{n1}^{(i)} e_{n1} + \cdots + \lambda_{n,m-1}^{(i)} e_{n,m-1} + \lambda_{nm}^{(i)} e_{nm}', \quad i = 1, \ldots, m, \\
e_{nm}' &= \begin{cases} 
\frac{1}{2} e_{nm} & \text{if } n \text{ is even}, \\
e_{nm} & \text{if } n \text{ is odd}.
\end{cases}
\end{align*}
\]

In other words,

\[
\cot \left( \frac{(2i-1)\pi}{2n} \right) = 2 \left[ \lambda_{n1}^{(i)} \sin \frac{\pi}{n} + \cdots + \lambda_{n,m-1}^{(i)} \sin \frac{(m-1)\pi}{n} + \theta_n \lambda_{nm}^{(i)} \sin \frac{m\pi}{n} \right],
\]

where

\[
\theta_n = \begin{cases} 
\frac{1}{2} & \text{if } n \text{ is even}, \\
1 & \text{if } n \text{ is odd}.
\end{cases}
\]

This is true by (12) when \( i = 1 \) (\( s_{n1} = s_n, \lambda_{n1}^{(1)} = \cdots = \lambda_{nm}^{(1)} = 1 \)) but remains generally open.

For example, let \( n = 15 \). Denoting \( s_k = s_{15,k} \) and \( e_k = e_{15,k} \) for brevity, we have [3, p. 17]

\[
\begin{align*}
\text{s}_1 &= e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 \\
\text{s}_2 &= e_3 + e_5 \\
\text{s}_3 &= e_5 \\
\text{s}_4 &= e_1 - e_2 + e_3 - e_4 + e_5 - e_6 + e_7 \\
\text{s}_5 &= -e_3 + e_6 \\
\text{s}_6 &= e_1 - e_2 + e_3 + e_4 - e_5 - e_6 + e_7 \\
\text{s}_7 &= e_1 + e_2 - e_3 - e_4 + e_5 + e_6 - e_7.
\end{align*}
\]
We study the zero coefficients in general. If and only if \( d = \gcd(n, 2i-1) > 1 \), then \( G_n \) "inherits" the chord
\[
s_{ni} = \cot \left( \frac{(2i - 1)\pi}{2n} \right)
\]
from \( G_d \). Then the chords of \( G_d \) are enough to express \( s_{ni} \), and the coefficients of the remaining chords are zero. Indeed, in our example,
\[
\begin{align*}
s_2 &= s_{15,2} = \cot \frac{3\pi}{30} = \cot \frac{\pi}{10} = 2 \left( \sin \frac{\pi}{5} + \sin \frac{2\pi}{5} \right), \\
s_3 &= s_{15,3} = \cot \frac{5\pi}{30} = \cot \frac{\pi}{6} = 2 \sin \frac{\pi}{3}, \\
s_5 &= s_{15,5} = \cot \frac{9\pi}{30} = \cot \frac{3\pi}{10} = 2 \left( -\sin \frac{\pi}{5} + \sin \frac{2\pi}{5} \right),
\end{align*}
\]
showing that \( s_3 \) is "inherited" from \( G_3 \), and \( s_2 \) and \( s_5 \) from \( G_5 \).

So we conjecture additionally that if and only if \( n \) is a prime or a power of 2, then each \( \lambda_{nk}^{(i)} \in \{ \pm 1 \} \). Mustonen [3] gives also other experimental results and conjectures about the structure of the three-dimensional array \( (\lambda_{nk}^{(i)}) \), and presents an efficient algorithm to compute these numbers.

6 Connections with OEIS sequences

The (lexicographically ordered) sequence \( (a_{mk}) \) is A053122 in OEIS. Its first six terms are \( a_{00} = 1 \), \( a_{10} = -2 \), \( a_{11} = 1 \), \( a_{20} = 3 \), \( a_{21} = -4 \), \( a_{22} = 1 \).

The OEIS sequence A132460 consists of the numbers
\[
t_{n0} = 1, \quad n = 0, 1, 2, \ldots, \\
t_{nk} = (-1)^k \left( \binom{n-k}{k} + \binom{n-k-1}{k-1} \right), \quad n = 2, 3, \ldots, k = 1, \ldots, m.
\]
The first six terms of its subsequence corresponding to odd values of \( n \) are \( t_{10} = 1 = b_{00} \), \( t_{30} = 1 = b_{11} \), \( t_{31} = -3 = b_{10} \), \( t_{50} = 1 = b_{22} \), \( t_{51} = -5 = b_{21} \), \( t_{52} = 5 = b_{20} \). In general, \( b_{mk} = t_{2m+1,m-k} \).

Also the characteristic polynomials of certain other tridiagonal matrices have connections with OEIS sequences. We study two of them.

Let \( \text{tridiag}(a, b, c) \) denote the tridiagonal matrix with main diagonal, sub-diagonal and super-diagonal entries those of vectors \( a \), \( b \) and \( c \), respectively, and denote \( x^{(k)} = x, \ldots, x, \) \( k \) copies. For \( m \geq 3 \), define
\[
C_m = \text{tridiag} \left( (2^{(m)})^{(m-2)}, (2^{(m-2)})^{(m-2)}, (2, (2^{(m-2)}))^{(m-2)} \right)
\]
and

\[ C_2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad C_1 = (2). \]

For \( m \geq 1 \), consider the polynomial

\[ C_m(x) = \det(xI_m - C_m) = x^m + c_{m,m-1}x^{m-1} + \cdots + c_{m1}x + c_{m0} \]

and define \( C_0(x) = 1, \ c_{00} = c_{mm} = 1 \). The sequence A140882 consists of the numbers \((-1)^m c_{mk}\). Since \( C_0(x) = 1, \ C_1(x) = x - 2, \ C_2(x) = x^2 - 4x, \ C_3(x) = x^3 - 6x^2 + 8x \), its first ten terms are 1, 2, \(-1\), 0, \(-4\), 1, 0, \(-8\), 6, \(-1\), as listed in [4].

We have \( x\tilde{A}_1(x) = x^2 - 4x = C_2(x) \) and \( x\tilde{A}_2(x) = x^3 - 6x^2 + 8x = C_3(x) \), and generally

\[ C_{m+1}(x) = x\tilde{A}_m(x) \]  \hspace{1cm} (19)

for all \( m \geq 1 \). This can be proved similarly to the proofs of Theorems 1 and 2.

By (8), a formula for A140882 is then obtained. By (19), (7) and (3),

\[ \text{spec} \ C_m = \text{spec} \ A_{m-2} \cup \{0,4\} = \left\{ 4 \sin^2 \frac{k\pi}{2m - 2} \bigg| k = 0, \ldots, m - 1 \right\} \]

for \( m \geq 3 \).

Finally, the sequence A136672 motivates us to study the polynomial

\[ f_{m+1}(x) = (x - 2)A_m(x) = x^{m+1} + f_{m+1,m}x^m + \cdots + f_{m+1,1}x + f_{m+1,0} \]  \hspace{1cm} (20)

and its connections with the matrix \( D_m \), defined by

\[ D_m = \text{tridiag}(\{2^{|m|}\}, \{-1\}^{m-2}, 0, \{-1\}^{m-1}) \]

if \( m \geq 3 \), and

\[ D_2 = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}, \quad D_1 = (2). \]

By Theorem 1,

\[ f_{m+1,k} = (-1)^{m-k+1}\left( \binom{m+k}{2k-1} + 2\binom{m+1+k}{2k+1} \right). \]  \hspace{1cm} (21)

For \( m \geq 1 \), consider the polynomial

\[ D_m(x) = \det(xI_m - D_m) = x^m + d_{m,m-1}x^{m-1} + \cdots + d_{m1}x + d_{m0} \]
and define \( D_0(x) = 1 \), \( d_{00} = d_{mm} = 1 \). The sequence A136672 consists of the numbers \((-1)^m d_{mk}\). We have \( D_0(x) = 1 \), \( D_1(x) = x - 2 \), \( D_2(x) = x^2 - 4x + 4 \), \( D_3(x) = x^3 - 6x^2 + 11x - 6 \). So its first ten terms are 1, 2, \(-1\), 4, \(-4\), 1, 6, \(-11\), 6, \(-1\), as listed in [4].

Since \( F_1(x) = x - 2 = D_1(x) \), \( F_2(x) = x^2 - 4x + 4 = D_2(x) \), and \( F_3(x) = x^3 - 6x^2 + 11x - 6 = D_3(x) \), it seems that

\[
D_m(x) = F_m(x)
\]

for all \( m \geq 1 \). This can be proved similarly to the previous proofs. By (21), a formula for A136672 follows. By (22), (20) and (3),

\[
spec D_m = spec A_{m-1} \cup \{2\} = \left\{ 4 \sin^2 \frac{k\pi}{2m} \mid k = 1, \ldots, m-1 \right\} \cup \{2\}
\]

for \( m \geq 2 \).

7 Conclusions and further questions

The squared chords of \( G_n \), excluding the diameter, are the zeros of a monic polynomial of degree \( \mu \) with integer coefficients. Including the diameter, the degree is \( m \).

The squared sum of all individual chords is the largest zero of a monic polynomial of degree \( m \) with integer coefficients. An equivalent fact is that the squared sum of all different (lengths of) chords but the diameter is halved, is a zero of such a polynomial. The zeros of this polynomial seem to be linear combinations of the chords with all coefficients 0 or \( \pm 1 \).

Lemma 1, stating that \( e_{n1}^2, \ldots, e_{n\mu}^2 \) are the eigenvalues of a tridiagonal matrix with integer entries, follows from certain properties of the Chebychev polynomials. So squared chords have interesting connections with these topics. But what about \( s_{n1}^2, \ldots, s_{nm}^2 \)? Are also they the eigenvalues of such a tridiagonal matrix? This question remains open.

The coefficients of the polynomial \( (x - e_{n1}^2) \cdots (x - e_{n\mu}^2) \) form an OEIS sequence, and so do also those of certain related polynomials. What about the coefficients of \( (x - s_{n1}^2) \cdots (x - s_{nm}^2) \)? Do also they form such a sequence? This question remains open, too.
Appendix: Proofs of (6) and (11)

Proof of (6)

\[(x - 2)P_m(x) - P_{m-1}(x)\]

\[= (x - 2) \sum_{k=0}^{m} (-1)^{m-k} \binom{m + 1 + k}{2k + 1} x^k - \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m + k}{2k + 1} x^k\]

\[-x^{m+1} + \sum_{k=0}^{m-1} (-1)^{m-k} \binom{m + 1 + k}{2k + 1} x^{k+1} - 2 \sum_{k=0}^{m} (-1)^{m-k} \binom{m + 1 + k}{2k + 1} x^k\]

\[-\sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m + k}{2k + 1} x^k\]

\[= x^{m+1} + \sum_{k=1}^{m} (-1)^{m+1-k} \binom{m + k}{2k - 1} x^k + 2 \sum_{k=0}^{m} (-1)^{m+1-k} \binom{m + 1 + k}{2k + 1} x^k\]

\[-\sum_{k=0}^{m-1} (-1)^{m+1-k} \binom{m + k}{2k + 1} x^k\]

\[= x^{m+1} - (\binom{2m}{2m - 1} + 2 \binom{2m + 1}{2m + 1}) x^m\]

\[+ \sum_{k=1}^{m-1} (-1)^{m+1-k} \left( \binom{m + k}{2k - 1} + 2 \binom{m + 1 + k}{2k + 1} - \binom{m + k}{2k + 1} \right) x^k\]

\[+ (-1)^{m+1} \left( 2 \binom{m + 1}{1} - \binom{m}{1} \right)\]

\[= x^{m+1} - (2m + 2)x^m + \sum_{k=1}^{m-1} (-1)^{m+1-k} \binom{m + 2 + k}{2k + 1} x^k + (-1)^{m+1}(m + 2)\]

\[= \sum_{k=0}^{m+1} (-1)^{m+1-k} \binom{m + 1 + 1 + k}{2k + 1} x^k = P_{m+1}(x).\]
Proof of (11)

\[(x - 3)P_m(x) - P_{m-1}(x)\]
\[= \cdots = x^{m+1} - \left(\binom{2m}{2m-1} + 3\binom{2m + 1}{2m + 1}\right)x^m\]
\[+ \sum_{k=1}^{m-1} (-1)^{m+1-k} \left(\binom{m+k}{2k-1} + 3\binom{m+1+k}{2k+1} - \binom{m+k}{2k+1}\right)x^k\]
\[+ (-1)^{m+1} \left(3\binom{m+1}{1} - \binom{m}{1}\right)\]
\[= x^{m+1} - (2m + 3)x^m + \sum_{k=1}^{m-1} (-1)^{m+1-k} \frac{2m + 3}{m - k + 1} \binom{m + 1 + k}{2k + 1} x^k\]
\[+ (-1)^{m+1}(2m + 3)\]
\[= x^{m+1} + \sum_{k=0}^{m} (-1)^{m+1-k} \frac{2(m+1)+1}{m+1-k} \binom{m+1+k}{2k+1} x^k = Q_{m+1}(x).\]

References


http://oeis.org/


http://mathworld.wolfram.com/Sine.html
Polynomials associated with regular polygons

    http://mathworld.wolfram.com/Tangent.html


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