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Some polynomials associated with regular polygons

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Abstract. Let G_n be a regular n-gon with unit circumradius, and $m = \lfloor \frac{n}{2} \rfloor$, $\mu = \lfloor \frac{n-1}{2} \rfloor$. Let the edges and diagonals of G_n be $e_{n1} < \cdots < e_{nm}$. We compute the coefficients of the polynomial

$$(x - e_{n1}^2) \cdots (x - e_{n\mu}^2).$$

They appear to form a well-known integer sequence, and we study certain related sequences, too. We also compute the coefficients of the polynomial

$$(x - s_{n1}^2) \cdots (x - s_{nm}^2),$$

where

$$s_{ni} = \cot \frac{(2i-1)\pi}{2n}, \quad i = 1, \ldots, m.$$

We interpret s_{n1} as the sum of all individual edges and diagonals of G_n divided by n. We also discuss the interpretation of $s_{n2}, \ldots, s_{nm}$, and present a conjecture on expressing $s_{n1}, \ldots, s_{nm}$ using $e_{n1}, \ldots, e_{nm}$.

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1 Introduction

Throughout, G_n is a regular n-gon with unit circumradius, and

$$m = \left\lfloor \frac{n}{2} \right\rfloor, \quad \mu = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Long time ago Kepler observed [2] that the squares of the edge and diagonals of G_7 are the zeros of the polynomial $x^3 - 7x^2 + 14x - 7$. This raises a general question: Are the squares of (the lengths of) the edge and diagonals of G_n, excluding the diameter, the zeros of a monic polynomial of degree μ with integer coefficients?

Yes, they are. This follows from Savio's and Suruyanarayan's [6] results, which, however, do not give the polynomial explicitly. We will do it in Section 2. A natural further question concerns the edge and diagonals themselves, instead of their squares. They are not zeros of a polynomial described above, but we will in Section 3 see that the squared sum of all individual edges and diagonals is the largest zero of a monic polynomial of degree m with integer coefficients. We will study geometric interpretation of the square roots of the other zeros in Section 4. In Section 5, we will present a conjecture on expressing these square roots as simple linear combinations of the edge and diagonals. We will in Section 6 notify that the coefficients of the first-mentioned polynomial form an OEIS [4] sequence, and also study OEIS sequences corresponding to certain related polynomials. Finally, we will complete our paper with conclusions and further questions in Section 7.

2 Squared chords

Let (the lengths of) the edge and diagonals of G_n be $e_{n1} < \cdots < e_{nm}$. Call them (the lengths of) the *chords*. Then

$$e_{nk} = 2 \sin \frac{k\pi}{n}, \quad k = 1, \ldots, m.$$

Our problem is to find the coefficients a_{mk} and b_{mk} of the polynomials

$$A_m(x) = (x - e^2_{n+2,1}) \cdots (x - e^2_{n+2,m}) = \\ x^m + a_{m,m-1}x^{m-1} + \cdots + a_{m1}x + a_{m0}, \quad (1)$$

where n is even, and

$$B_m(x) = (x - e^2_{n1}) \cdots (x - e^2_{nm}) = x^m + b_{m,m-1}x^{m-1} + \cdots + b_{m1}x + b_{m0}, \quad (2)$$

where n is odd. We solve it in two theorems. Mustonen [3] found them experimentally and sketched their proofs.

Let $\text{tridiag}_m(x, y)$ denote the symmetric tridiagonal $m \times m$ matrix with all main diagonal entries x and first super- and subdiagonal entries y. For $m \geq 2$, define

$$\mathbf{A}_m = \text{tridiag}_m(2, 1)$$

and

$$\mathbf{B}_m \text{ is as } \mathbf{A}_m \text{ but the } (m, m) \text{ entry equals } 3.$$

Also define $\mathbf{A}_1 = (2)$ and $\mathbf{B}_1 = (3)$. Denote by spec the (multi)set of eigenvalues.

Lemma 1 *For all* $m \geq 1$,

$$\text{spec}\,\mathbf{A}_m = \left\{4\sin^2 \frac{k\pi}{n+2} \,\middle|\, k = 1, \ldots, m\right\} = \{e_{n+2,1}^2, \ldots, e_{n+2,m}^2\}, \tag{3}$$
$$\text{spec}\,\mathbf{B}_m = \left\{4\sin^2 \frac{k\pi}{n} \,\middle|\, k = 1, \ldots, m\right\} = \{e_{n1}^2, \ldots, e_{nm}^2\}.$$

Proof. See [1, 5, 6]. □

Theorem 1 *In* (1),

$$a_{mk} = (-1)^{m-k}\binom{m+1+k}{2k+1}. \tag{4}$$

Proof. Denoting

$$P_m(x) = x^m + \sum_{k=0}^{m-1} (-1)^{m-k}\binom{m+1+k}{2k+1}x^k,$$

our claim is that

$$P_m(x) = A_m(x) \tag{5}$$

for all $m \geq 1$. Expanding $\det(x\mathbf{I}_m - \mathbf{A}_m)$ along the last row, we have

$$A_{m+1}(x) = (x-2)A_m(x) - A_{m-1}(x)$$

for all $m \geq 2$. Since

$$P_1(x) = x - 2 = A_1(x)$$

and

$$P_2(x) = x^2 - 4x + 3 = A_2(x),$$

the claim (5) follows by showing that

$$P_{m+1}(x) = (x-2)P_m(x) - P_{m-1}(x) \tag{6}$$

for all $m \geq 2$. Mustonen [3] did it by using Mathematica. We will do the computations algebraically in the appendix. □

The formula (4) yields $a_{mm} = 1$, consistently with the coefficient of x^m in (1). It also allows to define $a_{00} = 1$. The polynomial

$$\tilde{A}_{m+1}(x) = (x-4)A_m(x) = x^{m+1} + \alpha_{m+1,m}x^m + \cdots + \alpha_{m+1,1}x + \alpha_{m+1,0} \tag{7}$$

has $e^2_{n+2,m+1} = 4$ as the additional zero. By (4),

$$\alpha_{m+1,k} = (-1)^{m-k+1}\left(\binom{m+k}{2k-1} + 4\binom{m+1+k}{2k+1}\right). \tag{8}$$

(We define $\binom{n}{k} = 0$ if $k < 0$.)

Theorem 2 *In* (2),

$$b_{mk} = (-1)^{m-k}\frac{2m+1}{m-k}\binom{m+k}{2k+1} = (-1)^{m-k}\left(\binom{m+1+k}{2k+1} + \binom{m+k}{2k+1}\right). \tag{9}$$

Proof. The second equation follows from trivial computation. To show the first, denote

$$Q_m(x) = x^m + \sum_{k=0}^{m-1}(-1)^{m-k}\frac{2m+1}{m-k}\binom{m+k}{2k+1}x^k$$

and claim that

$$Q_m(x) = B_m(x) \tag{10}$$

for all $m \geq 1$. Expanding $\det(x\mathbf{I}_m - \mathbf{B}_m)$, we have

$$B_{m+1}(x) = (x-3)A_m(x) - A_{m-1}(x)$$

for all $m \geq 2$. Since

$$Q_1(x) = x - 3 = B_1(x)$$

and

$$Q_2(x) = x^2 - 5x + 5 = B_2(x),$$

the claim (10) follows by showing that

$$Q_{m+1}(x) = (x-3)P_m(x) - P_{m-1}(x) \tag{11}$$

for all $m \geq 2$. Mustonen [3] did also this by using Mathematica, and we will do the computations algebraically in the appendix. □

For $k = m$, the first expression in (9) is undefined but the second is defined. (We define $\binom{n}{k} = 0$ if $n < k$.) It gives $b_{mm} = 1$, the coefficient of x^m in (2). It also allows to define $b_{00} = 1$.

Corollary 1 *The sum of all individual squared chords of* G_n *is* n^2*. Their product is* n^n*.*

Proof. By Theorems 1 and 2 (or by [7, Eqs. (20) and (24)]), we obtain

$$\begin{aligned} e_{2m,1}^2 + \cdots + e_{2m,m-1}^2 &= -a_{m-1,m-2} = 2(m-1), \\ e_{2m+1,1}^2 + \cdots + e_{2m+1,m}^2 &= -b_{m,m-1} = 2m+1, \end{aligned}$$

and

$$\begin{aligned} e_{2m,1}^2 \cdots e_{2m,m-1}^2 &= (-1)^m a_{m-1,0} = m, \\ e_{2m+1,1}^2 \cdots e_{2m+1,m}^2 &= (-1)^m b_{m0} = 2m+1. \end{aligned}$$

Denoting by Σ_n the sum and by Π_n the product of all individual squared chords of G_n, we therefore have

$$\begin{aligned} \Sigma_{2m} &= 2m \cdot 2(m-1) + m \cdot 4 = (2m)^2, \\ \Sigma_{2m+1} &= (2m+1)(2m+1) = (2m+1)^2, \end{aligned}$$

and

$$\Pi_{2m} = m^{2m} 4^m = (2m)^{2m}, \quad \Pi_{2m+1} = (2m+1)^{2m+1}.$$

□

3 Sum of chords

The sum of all individual chords of G_n is

$$S_n = ns_n,$$

where

$$s_n = e_{n1} + \cdots + e_{n,m-1} + \tfrac{1}{2}e_{nm} = e_{n1} + \cdots + e_{n,m-1} + 1$$

if n is even, and

$$s_n = e_{n1} + \cdots + e_{nm}$$

if n is odd, is the sum of different (lengths of) chords but the diameter is halved.

Theorem 3 *For all* $n \geq 3$,

$$s_n = \cot\frac{\pi}{2n}.$$

Proof. We have [7, Eq. (21)]

$$\sum_{k=1}^{n-1} \sin\frac{k\pi}{n} = \cot\frac{\pi}{2n}. \tag{12}$$

If n is even, this implies

$$s_n = \sum_{k=1}^{m-1} 2\sin\frac{k\pi}{n} + \frac{1}{2}\cdot 2 = \sum_{k=1}^{m-1} \sin\frac{k\pi}{n} + 1 + \sum_{k=m+1}^{2m-1} \sin\frac{k\pi}{n} = \sum_{k=1}^{2m-1} \sin\frac{k\pi}{n} = \cot\frac{\pi}{2n}.$$

If n is odd, then

$$s_n = \sum_{k=1}^{m} 2\sin\frac{k\pi}{n} = \sum_{k=1}^{m} \sin\frac{k\pi}{n} + \sum_{k=m+1}^{2m} \sin\frac{k\pi}{n} = \sum_{k=1}^{2m} \sin\frac{k\pi}{n} = \cot\frac{\pi}{2n}.$$

□

Is s_n a zero of a monic polynomial of degree m with integer coefficients? Yes for $s_4 = \cot\frac{\pi}{8} = 1+\sqrt{2}$; it is a zero of x^2-2x-1. On the other hand, it is easy to see that $s_5 = \cot\frac{\pi}{10} = \sqrt{5+2\sqrt{5}}$ is not a zero of such a polynomial. But

$s_5^2 = 5+2\sqrt{5}$ is a zero of $x^2-10x+5$, and the other zero is $5-2\sqrt{5} = \cot^2\frac{3\pi}{10}$. Also $s_4^2 = 3+2\sqrt{2}$ has this property: it is a zero of x^2-6x+1, and the other zero is $3-2\sqrt{2} = \cot^2\frac{3\pi}{8}$.

Generally, denoting

$$s_{ni} = \cot\frac{(2i-1)\pi}{2n}, \quad i = 1,\dots,m,$$

this motivates us to study for even n the coefficients of the polynomial

$$U_m(x) = (x-s_{n1}^2)\cdots(x-s_{nm}^2) = x^m + u_{m,m-1}x^{m-1} + \cdots + u_{m1}x + u_{m0}, \quad (13)$$

and for odd n those of

$$V_m(x) = (x-s_{n1}^2)\cdots(x-s_{nm}^2) = x^m + v_{m,m-1}x^{m-1} + \cdots + v_{m1}x + v_{m0}. \quad (14)$$

We will see that they all are integers. The largest zero is $s_n^2 = s_{n1}^2$.

Mustonen [3] found the following theorem experimentally and also presented its proof. Yaglom and Yaglom [9, Eqs. (7) and (8)] formulated (16) differently.

Theorem 4 *In* (13),

$$u_{mk} = (-1)^k\binom{n}{2k}. \quad (15)$$

In (14),

$$v_{mk} = (-1)^k\binom{n}{2k+1}. \quad (16)$$

Proof. We have [10]

$$\cot nt = \frac{\sum_{k=0}^{m}(-1)^k\binom{n}{2k}\cot^{n-2k}t}{\sum_{k=0}^{m}(-1)^k\binom{n}{2k+1}\cot^{n-2k-1}t}. \quad (17)$$

Denote

$$t_i = \frac{(2i-1)\pi}{2n}, \quad i = 1,\dots,m.$$

Since $\cot nt_i = 0$, (17) yields

$$\sum_{k=0}^{m}(-1)^k\binom{n}{2k}\cot^{n-2k}t_i = 0. \quad (18)$$

First assume n even. The polynomial

$$\tilde{U}_m(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{n}{2k} x^k$$

is monic and has degree m. For all $i = 1, \ldots, m$,

$$\begin{aligned}\tilde{U}_m(s_{ni}^2) &= \sum_{k=0}^{m} (-1)^{m-k} \binom{2m}{2k} s_{ni}^{2k} = \sum_{l=0}^{m} (-1)^{l} \binom{2m}{2m-2l} s_{ni}^{2m-2l} \\ &= \sum_{l=0}^{m} (-1)^{l} \binom{2m}{2l} s_{ni}^{2m-2l} = \sum_{l=0}^{m} (-1)^{l} \binom{n}{2l} \cot^{n-2l} t_i = 0\end{aligned}$$

by (18). Hence

$$\tilde{U}_m(x) = (x - s_{n1}^2) \cdots (x - s_{nm}^2) = U_m(x),$$

and (15) follows.

Second, assume n odd. The polynomial

$$\tilde{V}_m(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{n}{2k+1} x^k$$

is monic and has degree m. For all $i = 1, \ldots, m$,

$$\begin{aligned}\tilde{V}_m(s_{ni}^2) &= \sum_{k=0}^{m} (-1)^{m-k} \binom{2m+1}{2k+1} s_{ni}^{2k} = \sum_{l=0}^{m} (-1)^{l} \binom{2m+1}{2m-2l+1} s_{ni}^{2m-2l} = \\ & s_{ni}^{-1} \sum_{l=0}^{m} (-1)^{l} \binom{2m+1}{2m-2l+1} s_{ni}^{2m+1-2l} = s_{ni}^{-1} \sum_{l=0}^{m} (-1)^{l} \binom{2m+1}{2l} s_{ni}^{2m+1-2l} \\ &= s_{ni}^{-1} \sum_{l=0}^{m} (-1)^{l} \binom{n}{2l} \cot^{n-2l} t_i = 0,\end{aligned}$$

again by (18). Hence

$$\tilde{V}_m(x) = (x - s_{n1}^2) \cdots (x - s_{nm}^2) = V_m(x),$$

and (16) follows. □

Corollary 2 *The number* s_n^2 *is the largest zero of the polynomial*

$$x^m + u_{m,m-1} n x^{m-1} + \cdots + u_{m1} n^{m-1} x + u_{m0} n^m$$

if n *is even, and that of*

$$x^m + v_{m,m-1} n x^{m-1} + \cdots + v_{m1} n^{m-1} x + v_{m0} n^m$$

if n *is odd.*

4 Interpreting $s_{n,m-k+1}$, $k = 1, \ldots, \lfloor \frac{n-1}{3} \rfloor$, n odd

The zeros of $A_m(x)$ and $B_m(x)$ describe the squared chords of G_{2m+2} and G_{2m+1}, respectively, excluding the diameter. The largest zero of $U_m(x)$, $s_{2m,1}^2 = s_{2m}^2$, and that of $V_m(x)$, $s_{2m+1,1}^2 = s_{2m+1}^2$, describe the squared sum of chords but halving the diameter. In other words, the sum of all individual chords of G_n is divided by n and the result is squared.

What about the other zeros?

Let the vertices of G_n be $P_0, \ldots, P_{n-1}$, where $P_k = (\cos \frac{k\pi}{n}, \sin \frac{k\pi}{n})$. Then $e_{nk} = P_0P_k = 2 \sin \frac{k\pi}{n}$, $k = 1, \ldots, m$. Since $P_0P_{n-k} = P_0P_k$, we define $e_{n,n-k} = e_{nk}$, $k = 1, \ldots, m$.

Fix n and denote $e_k = e_{nk}$ for brevity. Assume that $3k < n$; i.e., $k < \frac{n}{3}$. Then the line segments P_0P_{2k} and P_kP_{n-k} intersect; let Q_k be their intersection point and denote $x_k = P_0Q_k$. Because $\triangle Q_kP_0P_k \sim \triangle Q_kP_{2k}P_{n-k}$, we have

$$\frac{x_k}{e_{2k} - x_k} = \frac{e_k}{e_{3k}}.$$

Hence

$$x_k = \frac{e_k e_{2k}}{e_k + e_{3k}} = \frac{2 \sin \frac{k\pi}{n} \sin \frac{2k\pi}{n}}{\sin \frac{k\pi}{n} + \sin \frac{3k\pi}{n}} =$$
$$\frac{2 \sin \frac{k\pi}{n} \sin \frac{2k\pi}{n}}{\sin(\frac{2k\pi}{n} - \frac{k\pi}{n}) + \sin(\frac{2k\pi}{n} + \frac{k\pi}{n})} = \frac{\sin \frac{k\pi}{n} \sin \frac{2k\pi}{n}}{\sin \frac{2k\pi}{n} \cos \frac{k\pi}{n}} = \tan \frac{k\pi}{n}.$$

If n is odd, then

$$\tan \frac{k\pi}{n} = \cot \left(\frac{\pi}{2} - \frac{k\pi}{2m+1} \right) = \cot \frac{[2(m-k)+1]\pi}{2n} = s_{n,m-k+1}.$$

Thus $s_{n,m-k+1} = P_0Q_k$, $k = 1, \dots, \lfloor \frac{n-1}{3} \rfloor$. In other words, the $\lfloor \frac{n-1}{3} \rfloor$ smallest zeros of $V_m(x)$ are the squared line segments P_0Q_k, $k = 1, \dots, \lfloor \frac{n-1}{3} \rfloor$. Mustonen [3] found this experimentally. The largest zero is already interpreted, but the interpretation of the rest of zeros remains open. For some experimental observations, see [3]. Interpretation of the zeros of $U_m(x)$, except the largest, remains open, too.

5 Expressing $s_{n1}, \dots, s_{nm}$ using $e_{n1}, \dots, e_{nm}$

Mustonen's [3] experiments make conjecture that, given n, there are numbers $\lambda_{nk}^{(i)} \in \{0, \pm 1\}$, $i, k = 1, \dots, m$, such that

$$s_{ni} = \lambda_{n1}^{(i)} e_{n1} + \cdots + \lambda_{n,m-1}^{(i)} e_{n,m-1} + \lambda_{nm}^{(i)} e'_{nm}, \quad i = 1, \dots, m,$$

where

$$e'_{nm} = \begin{cases} \frac{1}{2} e_{nm} & \text{if } n \text{ is even,} \\ e_{nm} & \text{if } n \text{ is odd.} \end{cases}$$

In other words,

$$\cot \frac{(2i-1)\pi}{2n} = 2\Big[\lambda_{n1}^{(i)} \sin \frac{\pi}{n} + \cdots + \lambda_{n,m-1}^{(i)} \sin \frac{(m-1)\pi}{n} + \theta_n \lambda_{nm}^{(i)} \sin \frac{m\pi}{n}\Big],$$

where

$$\theta_n = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

This is true by (12) when $i = 1$ ($s_{n1} = s_n$, $\lambda_{n1}^{(1)} = \cdots = \lambda_{nm}^{(1)} = 1$) but remains generally open.

For example, let $n = 15$. Denoting $s_k = s_{15,k}$ and $e_k = e_{15,k}$ for brevity, we have [3, p. 17]

$$\begin{array}{lllllllll}
s_1 = & e_1 + & e_2 + & e_3 + & e_4 + & e_5 + & e_6 + & e_7 \\
s_2 = & & & e_3 + & & & e_6 & \\
s_3 = & & & & & e_5 & & \\
s_4 = & e_1 - & e_2 + & e_3 - & e_4 + & e_5 - & e_6 + & e_7 \\
s_5 = & & & -e_3 + & & & e_6 & \\
s_6 = & e_1 - & e_2 + & e_3 + & e_4 - & e_5 + & e_6 - & e_7 \\
s_7 = & e_1 + & e_2 - & e_3 - & e_4 + & e_5 + & e_6 - & e_7.
\end{array}$$

We study the zero coefficients in general. If and only if $d = \gcd(n, 2i{-}1) > 1$, then G_n "inherits" the chord

$$s_{ni} = \cot\frac{(2i-1)\pi}{2n}$$

from G_d. Then the chords of G_d are enough to express s_{ni}, and the coefficients of the remaining chords are zero. Indeed, in our example,

$$s_2 = s_{15,2} = \cot\frac{3\pi}{30} = \cot\frac{\pi}{10} = 2\Big(\sin\frac{\pi}{5} + \sin\frac{2\pi}{5}\Big),$$
$$s_3 = s_{15,3} = \cot\frac{5\pi}{30} = \cot\frac{\pi}{6} = 2\sin\frac{\pi}{3},$$
$$s_5 = s_{15,5} = \cot\frac{9\pi}{30} = \cot\frac{3\pi}{10} = 2\Big(-\sin\frac{\pi}{5} + \sin\frac{2\pi}{5}\Big),$$

showing that s_3 is "inherited" from G_3, and s_2 and s_5 from G_5.

So we conjecture additionally that if and only if n is a prime or a power of 2, then each $\lambda_{nk}^{(i)} \in \{\pm 1\}$. Mustonen [3] gives also other experimental results and conjectures about the structure of the three-dimensional array $(\lambda_{nk}^{(i)})$, and presents an efficient algorithm to compute these numbers.

6 Connections with OEIS sequences

The (lexicographically ordered) sequence (a_{mk}) is A053122 in OEIS. Its first six terms are $a_{00} = 1$, $a_{10} = -2$, $a_{11} = 1$, $a_{20} = 3$, $a_{21} = -4$, $a_{22} = 1$.

The OEIS sequence A132460 consists of the numbers

$$t_{n0} = 1, \quad n = 0, 1, 2, \ldots,$$
$$t_{nk} = (-1)^k(\binom{n-k}{k} + \binom{n-k-1}{k-1}), \quad n = 2, 3, \ldots,\; k = 1, \ldots, m.$$

The first six terms of its subsequence corresponding to odd values of n are $t_{10} = 1 = b_{00}$, $t_{30} = 1 = b_{11}$, $t_{31} = -3 = b_{10}$, $t_{50} = 1 = b_{22}$, $t_{51} = -5 = b_{21}$, $t_{52} = 5 = b_{20}$. In general, $b_{mk} = t_{2m+1,m-k}$.

Also the characteristic polynomials of certain other tridiagonal matrices have connections with OEIS sequences. We study two of them.

Let $\mathrm{tridiag}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ denote the tridiagonal matrix with main diagonal, subdiagonal and superdiagonal entries those of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, and denote $x^{(k)} = x, \ldots, x$, k copies. For $m \geq 3$, define

$$\mathbf{C}_m = \mathrm{tridiag}\,((2^{(m)}), ((-1)^{(m-2)}, -2), (-2, (-1)^{(m-2)}))$$

and

$$\mathbf{C}_2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad \mathbf{C}_1 = (2).$$

For $m \geq 1$, consider the polynomial

$$C_m(x) = \det(x\mathbf{I}_m - \mathbf{C}_m) = x^m + c_{m,m-1}x^{m-1} + \cdots + c_{m1}x + c_{m0}$$

and define $C_0(x) = 1$, $c_{00} = c_{mm} = 1$. The sequence A140882 consists of the numbers $(-1)^m c_{mk}$. Since $C_0(x) = 1$, $C_1(x) = x - 2$, $C_2(x) = x^2 - 4x$, $C_3(x) = x^3 - 6x^2 + 8x$, its first ten terms are $1, 2, -1, 0, -4, 1, 0, -8, 6, -1$, as listed in [4].

We have $x\tilde{A}_1(x) = x^2 - 4x = C_2(x)$ and $x\tilde{A}_2(x) = x^3 - 6x^2 + 8x = C_3(x)$, and generally

$$C_{m+1}(x) = x\tilde{A}_m(x) \tag{19}$$

for all $m \geq 1$. This can be proved similarly to the proofs of Theorems 1 and 2. By (8), a formula for A140882 is then obtained. By (19), (7) and (3),

$$\operatorname{spec}\mathbf{C}_m = \operatorname{spec}\mathbf{A}_{m-2} \cup \{0, 4\} = \left\{4\sin^2\frac{k\pi}{2m-2} \,\middle|\, k = 0, \ldots, m-1\right\}$$

for $m \geq 3$.

Finally, the sequence A136672 motivates us to study the polynomial

$$F_{m+1}(x) = (x-2)A_m(x) = x^{m+1} + f_{m+1,m}x^m + \cdots + f_{m+1,1}x + f_{m+1,0} \tag{20}$$

and its connections with the matrix $\mathbf{D}_m$, defined by

$$\mathbf{D}_m = \operatorname{tridiag}((2^{(m)}), ((-1)^{(m-2)}, 0), ((-1)^{(m-1)}))$$

if $m \geq 3$, and

$$\mathbf{D}_2 = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{D}_1 = (2).$$

By Theorem 1,

$$f_{m+1,k} = (-1)^{m-k+1}\left(\binom{m+k}{2k-1} + 2\binom{m+1+k}{2k+1}\right). \tag{21}$$

For $m \geq 1$, consider the polynomial

$$D_m(x) = \det(x\mathbf{I}_m - \mathbf{D}_m) = x^m + d_{m,m-1}x^{m-1} + \cdots + d_{m1}x + d_{m0}$$

and define $D_0(x) = 1$, $d_{00} = d_{mm} = 1$. The sequence A136672 consists of the numbers $(-1)^m d_{mk}$. We have $D_0(x) = 1$, $D_1(x) = x - 2$, $D_2(x) = x^2 - 4x + 4$, $D_3(x) = x^3 - 6x^2 + 11x - 6$. So its first ten terms are $1, 2, -1, 4, -4, 1, 6, -11, 6, -1$, as listed in [4].

Since $F_1(x) = x - 2 = D_1(x)$, $F_2(x) = x^2 - 4x + 4 = D_2(x)$, and $F_3(x) = x^3 - 6x^2 + 11x - 6 = D_3(x)$, it seems that

$$D_m(x) = F_m(x) \tag{22}$$

for all $m \geq 1$. This can be proved similarly to the previous proofs. By (21), a formula for A136672 follows. By (22), (20) and (3),

$$\operatorname{spec} \mathbf{D}_m = \operatorname{spec} \mathbf{A}_{m-1} \cup \{2\} = \left\{ 4 \sin^2 \frac{k\pi}{2m} \,\middle|\, k = 1, \ldots, m-1 \right\} \cup \{2\}$$

for $m \geq 2$.

7 Conclusions and further questions

The squared chords of G_n, excluding the diameter, are the zeros of a monic polynomial of degree μ with integer coefficients. Including the diameter, the degree is m.

The squared sum of all individual chords is the largest zero of a monic polynomial of degree m with integer coefficients. An equivalent fact is that the squared sum of all different (lengths of) chords but the diameter is halved, is a zero of such a polynomial. The zeros of this polynomial seem to be linear combinations of the chords with all coefficients 0 or ± 1.

Lemma 1, stating that $e_{n1}^2, \ldots, e_{n\mu}^2$ are the eigenvalues of a tridiagonal matrix with integer entries, follows from certain properties of the Chebychev polynomials. So squared chords have interesting connections with these topics. But what about $s_{n1}^2, \ldots, s_{nm}^2$? Are also they the eigenvalues of such a tridiagonal matrix? This question remains open.

The coefficients of the polynomial $(x - e_{n1}^2) \cdots (x - e_{n\mu}^2)$ form an OEIS sequence, and so do also those of certain related polynomials. What about the coefficients of $(x - s_{n1}^2) \cdots (x - s_{nm}^2)$? Do also they form such a sequence? This question remains open, too.

Appendix: Proofs of (6) and (11)

Proof of (6)

$$
\begin{aligned}
&(x-2)P_m(x) - P_{m-1}(x) \\
&= (x-2)\sum_{k=0}^{m}(-1)^{m-k}\binom{m+1+k}{2k+1}x^k - \sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m+k}{2k+1}x^k \\
&- x^{m+1} + \sum_{k=0}^{m-1}(-1)^{m-k}\binom{m+1+k}{2k+1}x^{k+1} - 2\sum_{k=0}^{m}(-1)^{m-k}\binom{m+1+k}{2k+1}x^k \\
&\quad - \sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m+k}{2k+1}x^k \\
&= x^{m+1} + \sum_{k=1}^{m}(-1)^{m+1-k}\binom{m+k}{2k-1}x^k + 2\sum_{k=0}^{m}(-1)^{m+1-k}\binom{m+1+k}{2k+1}x^k \\
&\quad - \sum_{k=0}^{m-1}(-1)^{m+1-k}\binom{m+k}{2k+1}x^k \\
&= x^{m+1} - \left(\binom{2m}{2m-1} + 2\binom{2m+1}{2m+1}\right)x^m \\
&\quad + \sum_{k=1}^{m-1}(-1)^{m+1-k}\left(\binom{m+k}{2k-1} + 2\binom{m+1+k}{2k+1} - \binom{m+k}{2k+1}\right)x^k \\
&\quad + (-1)^{m+1}\left(2\binom{m+1}{1} - \binom{m}{1}\right) \\
&= x^{m+1} - (2m+2)x^m + \sum_{k=1}^{m-1}(-1)^{m+1-k}\binom{m+2+k}{2k+1}x^k + (-1)^{m+1}(m+2) \\
&= \sum_{k=0}^{m+1}(-1)^{m+1-k}\binom{m+1+1+k}{2k+1}x^k = P_{m+1}(x).
\end{aligned}
$$

Proof of (11)

$$\begin{aligned}
&(x-3)P_m(x) - P_{m-1}(x) \\
&= \cdots = x^{m+1} - \left(\binom{2m}{2m-1} + 3\binom{2m+1}{2m+1}\right) x^m \\
&\quad + \sum_{k=1}^{m-1} (-1)^{m+1-k} \left(\binom{m+k}{2k-1} + 3\binom{m+1+k}{2k+1} - \binom{m+k}{2k+1}\right) x^k \\
&+ (-1)^{m+1} \left(3\binom{m+1}{1} - \binom{m}{1}\right) \\
&= x^{m+1} - (2m+3)x^m + \sum_{k=1}^{m-1} (-1)^{m+1-k} \frac{2m+3}{m-k+1} \binom{m+1+k}{2k+1} x^k \\
&\quad + (-1)^{m+1}(2m+3) \\
&= x^{m+1} + \sum_{k=0}^{m} (-1)^{m+1-k} \frac{2(m+1)+1}{m+1-k} \binom{m+1+k}{2k+1} x^k = Q_{m+1}(x).
\end{aligned}$$

References

[1] N. D. Cahill, J. R. D'Errico, J. P. Spence, Complex factorization of the Fibonacci and Lucas numbers, *Fibonacci Quart.*, **41** (2003), 13–19.

[2] H. M. S. Coxeter, *Regular Convex Polytopes*, Cambridge U. Pr., 1974.

[3] S. Mustonen, Lengths of edges and diagonals and sums of them in regular polygons as roots of algebraic equations, 2013, 44 pp.

`http://www.survo.fi/papers/Roots2013.pdf`

[4] *The On-Line Encyclopedia of Integer Sequences (OEIS).*

`http://oeis.org/`

[5] D. E. Rutherford, Some continuant determinants arising in physics and chemistry, I, *Proc. Royal Soc. Edinburgh*, **62A** (1947), 229–236.

[6] D. Y. Savio, E. R. Suryanarayan, Chebychev polynomials and regular polygons, *Amer. Math. Monthly*, **100** (1993), 657–661.

[7] E. W. Weisstein, Sine, *MathWorld –A Wolfram Web Resource.*

`http://mathworld.wolfram.com/Sine.html`

[8] E. W. Weisstein, Tangent, *MathWorld –A Wolfram Web Resource*.
`http://mathworld.wolfram.com/Tangent.html`

[9] А. М. Яглом, И. М. Яглом, Элементарный вывод формул Валлиса, Лейбница и Эйлера для числа π, *Успехи Матем. Наук*, **8** (1953), 181–187. (A. M. Yaglom, I. M. Yaglom, An elementary derivation of the Wallis, Leibniz and Euler formulas for the number π, *Uspekhi Matem. Nauk*, **8** (1953), 181–187.)

[10] `http://functions.wolfram.com/ElementaryFunctions/Cot/27/01/0002/`

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