'SIMPLE CONSTRUCTIONS' OF REGULAR N-SIDED POLYGONS AT ANY GIVEN ACCURACY

SEPPO MUSTONEN

Abstract. Consider n-sided polygons inscribed in a circle. The obvious fact that among all such polygons the regular one has the greatest area is proved by a trivial geometric argument. This result leads to an sequence of simple constructions where an originally irregular polygon grows towards a regular one at any given accuracy. It is also possible to improve approximate constructions for regular polygons like a heptagon where a traditional ruler and compass construction is impossible. Corresponding results are shown also for circumscribed polygons. The constructions are illustrated by 'live' demonstrations created as applications the Survo system [5].

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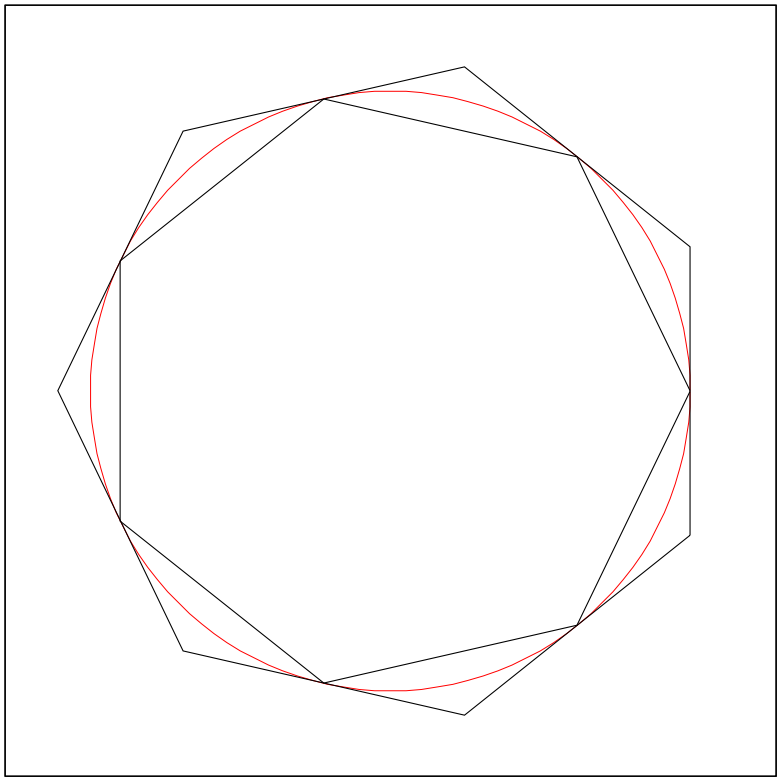

Figure 1. Inscribed and circuscribed regular heptagons

1. Introduction

When studying mathematics in the University of Helsinki, I attended a course on "calculus of several variables" in 1959. The course was based on a brilliant textbook by Ernst Lindelöf. In a chapter related to extreme values of functions of several variables, one of exercises was: "Prove that among all n-sided polygons located inside a circle the regular one has the largest area." It was obvious that this exercise was meant to be solved according to methods presented in the textbook. However, I got an idea that this problem does not necessarily require any information about "higher analysis". My solution was based on a direct observation that if the largest insribed n-sided polygon were irregular, its area could be increased at a pair of unequal adjacent sides of the polygon by moving the corresponding vertex to the middle of the arc of between the utmost vertices of the pair of sides.

Before going to the main topic, the approach used in Survo graphics is illustrated as Fig. 1 and a setup created for making it.

```
-------------------------------------------------------------------
  1 *SAVE N_GON_RAW / n-gons by direct PLOT schemes
  2 *LOAD INDEX
  3 * *GLOBAL* n=7 pi=3.141592653589793
  4 *
  5 *SCALE=-1.2,0,1.2 SIZE=1200,1200 XDIV=1,28,1 YDIV=1,28,1 HEADER=
  6 *XLABEL= YLABEL= FRAME=3
  7 * /ACTIVATE +  / This command activates all commands with '+'
  8 *.............................................................
  9 *Circle with unit radius:
 10+ PLOT X(t)=R*cos(t),Y(t)=R*sin(t) / DEVICE=PS,Circle1.ps R=1
 11*t=[line_width(0.24)][RED],0,2*pi,pi/60
 12*.............................................................
 13*Inscribed polygon: (using step 2*pi/n in plotting of a 'circle')
 14+ PLOT X(t)=R*cos(t),Y(t)=R*sin(t) / DEVICE=PS,PolygonI1.ps R=1
 15*t=[line_width(0.48)],0,2*pi,2*pi/n
 16*.............................................................
 17*Circumscribed polygon:
 18+ PLOT X(t)=R*cos(t),Y(t)=R*sin(t) / DEVICE=PS,PolygonC1.ps
 19*t=[line_width(0.48)],-pi/n,2*pi-pi/n,2*pi/n
 20*Actually plotted as an inscribed polygon for an invisible circle
 21*with radius R=1/cos(pi/n)
 22*Phase shift by -pi/n so that tangent points coincide with vertices
 23*of the inscribed polygon
 24*.............................................................
 25*Combining PostScript files generated by PLOT commands:
 26+ EPS JOIN Pict1,Circle1,PolygonI1,PolygonC1
 27+ EPS Pict1.ps Pict1.eps / convert to eps format
 28*
-------------------------------------------------------------------
```

Above is a snapshot from a Survo *edit field* containing all ingredients for making Fig. 1.
It is divided into 5 *subfields* separated by dotted lines. The first one is a *global*

subfield (see `*GLOBAL*` on line 3) giving common background information for other subfields. For example, `n=7` on the same line gives the number of vertices of polygons to be drawn; thus this is a general setup for plotting any n-gons and any interer, say 3,4,... may be selected instead of 7.
When the `/ACTIVATE +` command on line 7 is activated, it causes all commands having '`+`' in the *control column* (before ordinary text of a line) to be activated and then the `PLOT` commands on lines 10,14, and 18 draw a circle, an inscribed heptagon, and a circumscribed heptagon, respectively. These graphs are not shown immediately but saved in PostScript files `Circle1.ps`, `PolygonI1.ps`, and `PolygonC1.ps` as indicated by `DEVICE` *specifications* in each subfield.
Finally, these PostScript files are combined and converted into eps format by commands on lines 26,27.
Thus all this takes place automatically as a result of a single activation and the contents of the file `Pict1.eps` is shown in Fig. 1 by ordinary means of LaTeX.

The figures related to geometric constructions on the following pages are created by using the `GEOM` program module [3] of the Survo system [5].

2. Inscribed polygons

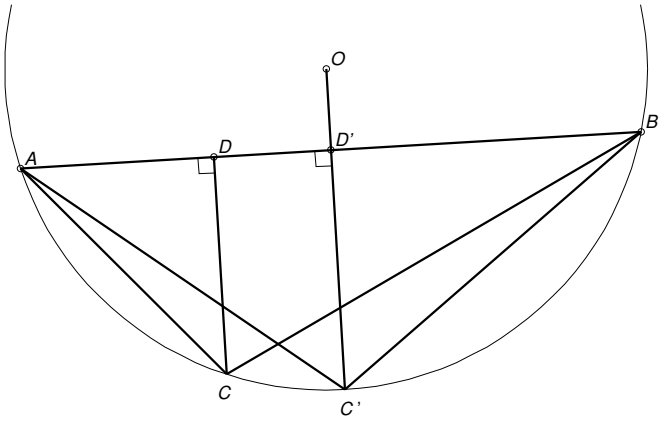

Figure 2

The fact that the regular n-sided polygon has the largest area is proved as follows:
Assume on the contrary that some non-regular n-gon has the largest area.
Then it must have at least two adjacent sides of different length. Consider the triangle consisting of those sides and the chord between their furthest end points, say A and B (in Fig. 2) Let the common point of sides be C and draw a perpendicular line from C to AB and let it meet AB at point D. Denote the length of the line segment CD by h and the length of AB by a. Then the area of the triangle is $ah/2$. Next draw a perpendicular for AB from the midpoint (say D') of AB and let C' be the point where it meets the arc ACB. Denote length of $C'D'$ by h'. Since the adjacent sides were unequal, h' is longer than h. If the vertex C is moved to C', the area of the triangle (now $ah'/2$) increases and so does the total area on the n-gon. Thus no irregular n-gon cannot have the largest area.

It is also interesting to see how much the area of the n-gon is grown by the above 'smoothing' construction. In Fig. 3 let $\angle BOC' = \alpha$ and $\angle COC' = \beta$. Then we have $\angle BAC = (\alpha + \beta)/2$ as an inscribed angle corresponding to the central angle $\angle BOC = \alpha + \beta$ and similarly $\angle ABC = (\alpha - \beta)/2$. The length of AB is $2\sin(\alpha)$. The area for a triangle with angles x, y and side a between them is given by the ASA formula

$$\text{Area}(x, y, a) = \tan(x)\tan(y)/(\tan(x) + \tan(y))a^2/2. \tag{1}$$

Then the area $|ABC|$ is obtained from this formula by selecting $x = (\alpha + \beta)/2$, $y = (\alpha - \beta)/2$, and $a = 2\sin(\alpha)$. Similarly, the area $|ABC'|$ is obtained from (1) by setting $\beta = 0$, i.e. by selecting $x = y = \alpha/2$ and $a = 2\sin(\alpha)$.

The difference $d_{in}(\alpha, \beta) = |ABC'| - |ABC|$ is computed, for example, by using Mathematica from Survo as follows:

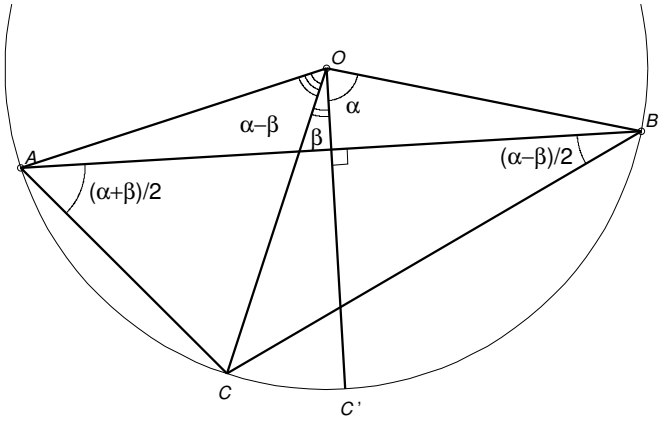

FIGURE 3

```
-----------------------------------------------------------------------
     *Saving Mathematica code in a text file:
     *SAVEW CUR+1,E,K.TXT
     *Area[x_,y_,z_]:=Tan[x]*Tan[y]/(Tan[x]+Tan[y])*z^2/2;
     *k1=Area[(a+b)/2,(a-b)/2,2*Sin[a]];
     *k2=Area[a/2,a/2,2*Sin[a]];
     *InputForm[Simplify[k2-k1]]
     E
     *Calling Mathematica to execute the code by a Survo macro /MATH:
     */MATH K.TXT
     *In[2]:= Area[x_,y_,z_]:=Tan[x]*Tan[y]/(Tan[x]+Tan[y])*z^2/2;
     *In[3]:= k1=Area[(a+b)/2,(a-b)/2,2*Sin[a]];
     *In[4]:= k2=Area[a/2,a/2,2*Sin[a]];
     *In[5]:= InputForm[Simplify[k2-k1]]
     *Out[5]//InputForm= 2*Sin[a]*Sin[b/2]^2
     *
-----------------------------------------------------------------------
```

Thus

$$d_{in}(\alpha,\beta) = 2\sin(\alpha)\sin(\beta/2)^2 \tag{2}$$

and since α is acute, $d_{in}(\alpha,\beta)$ is positive when $\beta > 0$.

By using this splitting construction repeatedly, any inscribed irregular n-gon can be grown towards a regular n-gon. To demonstrate this method, I have created a Survo macro, *sucro* `/NGON-IN`.

For example, activation of `/NGON-IN 7,20130408` in the Survo edit field leads to working with a heptagon (7-gon) by starting from a random inscribed irregular heptagon determined by a seed number (in this case 20130408) for a random number generator. The vertices will then be scattered randomly over the circumference.

`/NGON-IN` applies the splitting construction to the vertex which gives the maximum growth for area of the polygon according to the formula (2). At any step it is displayed in percentages how close the area of the current polygon is to that of the regular case.

A live demonstration of this example can be seen as a flash application
`http://www.survo.fi/flash/heptagon.html`
A series of snapshots is shown in Fig. 4.

The actual growth of the area and the vertex moved in the current round are presented in the following table:

```
-----------------------------------------------------------------
Round Area              Growth             Vertex
    0 2.34224245695511  -                  -
    1 2.646652520669050 0.304410063713938  1
    2 2.662569353150985 0.015916832481935  2
    3 2.691783224533356 0.029213871382371  3
    4 2.700487702444515 0.008704477911159  2
    5 2.713372525909583 0.012884823465068  1
    6 2.720503335643904 0.007130809734321  5
    7 2.726722085404965 0.006218749761061  4
-----------------------------------------------------------------
```

The growth typically becomes smaller in consecutive steps but not necessarily. For example, in the step from Round 2 to Round 3 it is larger than in the previous one, since then it becomes possible better than earlier to extend a short side 'around 9 o'clock'. In the previous round the vertex 'around 11 o'clock' has been moved further away from that short side.
It is obvious that in this way an approximation of a regular n-gon can be reached at any given accuracy by a finite number of steps.
The details about `/NGON-IN` are described in Appendix (not ready).

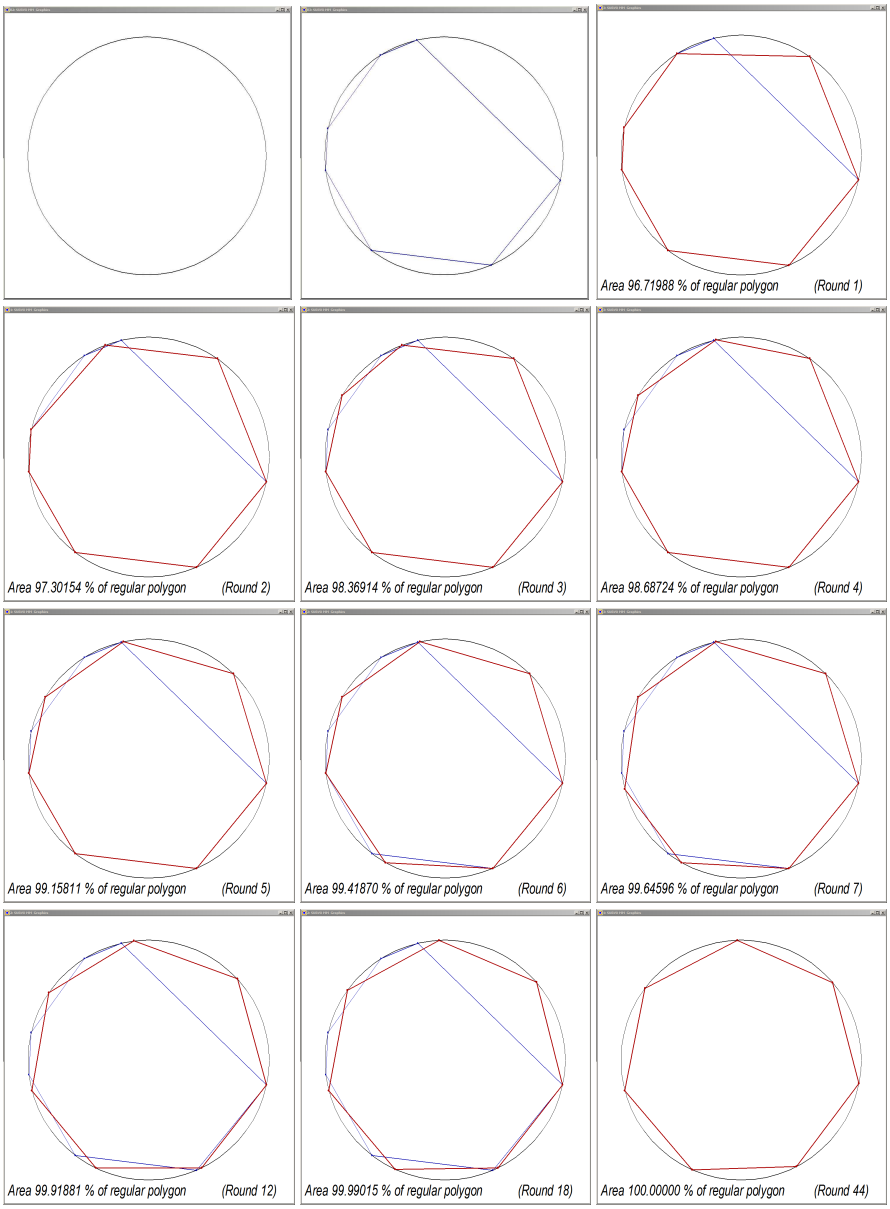

FIGURE 4

3. CIRCUMSCRIBED POLYGONS

The fact that the regular n-sided polygon has the smallest area is proved as follows:
Assume on the contrary that some non-regular n-gon has the smallest area.
Then it must have at least two adjacent vertices corresponding to different angles of corners like D and E in Fig. 5.
There C is a tangent point of the side DE of the n-gon. Let A and B be tangent

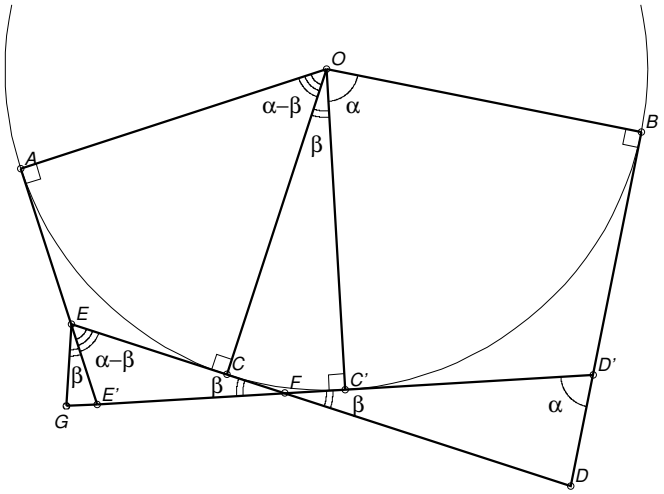

FIGURE 5

points of the neighbour sides of the n-gon. In this case we have $AE < BD$. It will be seen that area of n-gon is diminished by drawing a bisector OC' of $\angle AOB$ so that the side DE is replaced by $D'E'$ going through a new tangent point C'. Similarly AE is replaced by AE' and BD by BD'. Let F be the intersection point of DE and $D'E'$.

In the new situation triangle $\Delta FDD'$ is replaced by $\Delta FEE'$ as parts of the area inside the n-gon. Thus it should be shown that $|FD'D| > |FEE'|$.

Denote $\angle BOC' = \alpha$ and $\angle C'OC = \beta$ and observe that

$$0 < \beta < \alpha < \pi/2. \tag{3}$$

Then we have $\angle AOC = \alpha - \beta$ and, due to orthogonalities, $\angle FD'D = \alpha, \angle FEE' = \alpha - \beta$, and $\angle DFD' = \angle EFE' = \beta$.

Now extend $\Delta FEE'$ so that G is a point where line $D'E'$ and the line from E intersect so that $\angle GEE' = \beta$. Then $\Delta FEG \cong \Delta FD'D$. Now

$EF = EC + CF = \tan((\alpha - \beta)/2) + \tan(\beta/2)$ and

$D'F = D'C' + C'F = \tan(\alpha/2) + \tan(\beta/2)$

which implies that $D'F > EF$ and $|FD'D| > |FEG| > |FEE'|$.

The difference $d_{out}(\alpha,\beta) = |FD'D| - |FEE'|$ is obtained, again, by using Mathematica from Survo as follows:

```
-------------------------------------------------------------------
     *Saving Mathematica code in a text file:
     *SAVEW CUR+1,E,K.TXT
     *area[x_,y_,z_] := Tan[x]*Tan[y]/(Tan[x]+Tan[y])*z^2/2;
     *k1=area[a,b,Tan[a/2]+Tan[b/2]];
     *k2=area[a-b,b,Tan[(a-b)/2]+Tan[b/2]];
     *InputForm[Simplify[k1 - k2]]
     E
     *Calling Mathematica to execute the code by a Survo macro /MATH:
     */MATH K.TXT
     *In[2]:= area[x_,y_,z_] := Tan[x]*Tan[y]/(Tan[x]+Tan[y])*z^2/2;
     *In[3]:= k1=area[a,b,Tan[a/2]+Tan[b/2]];
     *In[4]:= k2=area[a-b,b,Tan[(a-b)/2]+Tan[b/2]];
     *In[5]:= InputForm[Simplify[k1 - k2]]
     *Out[5]//InputForm= 2*Sec[(a - b)/2]*Sec[(a + b)/2]*Sin[b/2]^2*Tan[a/2]
     *
-------------------------------------------------------------------
```

Thus the area is diminished by

$$d_{out}(\alpha,\beta) = 2\frac{\sin(\beta/2)^2\tan(\alpha/2)}{\cos((\alpha-\beta)/2)\cos((\alpha+\beta)/2)} \tag{4}$$

which is positive according to (3).

By using this splitting construction repeatedly, any circumscribed irregular n-gon can be reduced towards a regular n-gon. To demonstrate this method, I have created a Survo macro, *sucro* `/NGON-OUT`.

For example, activation of `/NGON-OUT 5,20130322` in the Survo edit field leads to working with a pentagon (5-gon) by starting from a random circumscribed irregular pentagon determined by a seed number (in this case 20130322) for a random number generator. In this case the tangent points of the sides of the polygon will be scattered randomly over the circumference.
`/NGON-OUT` applies the splitting construction to the vertex which gives the maximum decrease for the area of the polygon according to the formula (4). At any step it is displayed in percentages how close the area of the current polygon is to that of the regular case.
A live demonstration of this example can be seen as a flash application
`http://www.survo.fi/flash/pentagon.html`
A series of snapshots is shown in Fig. 6.

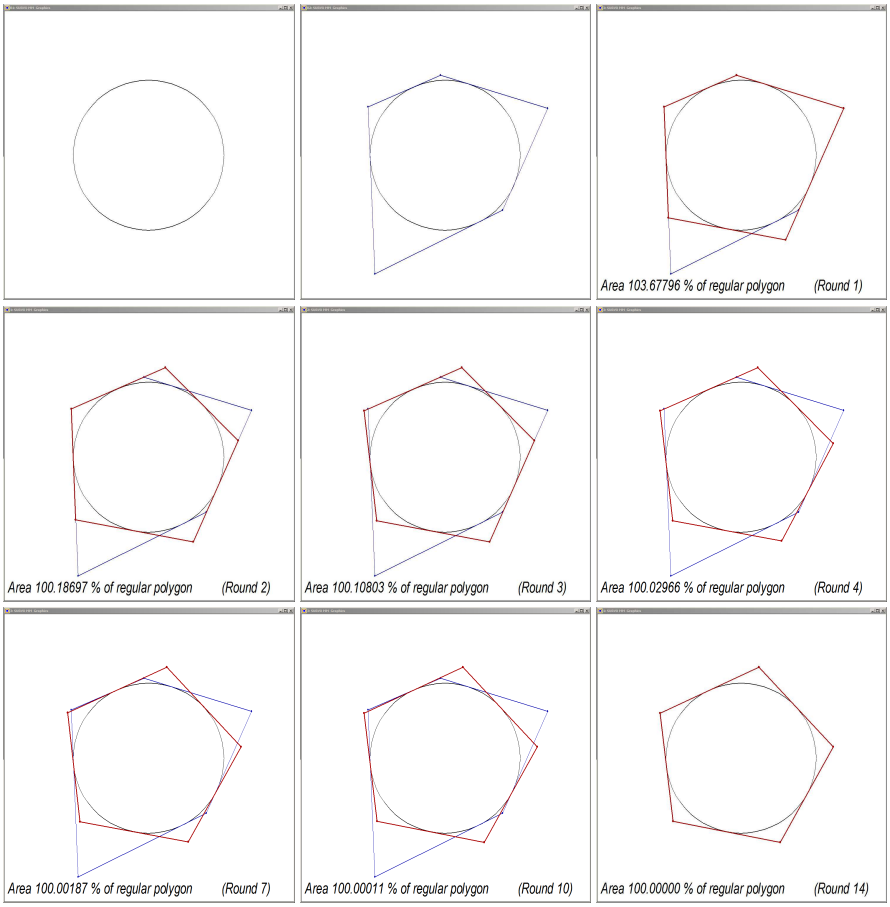

FIGURE 6

It is obvious that in this way an approximation of a regular n-gon can be reached at any given accuracy by a finite number of steps.
The details about `/NGON-OUT` are described in Appendix (not ready).

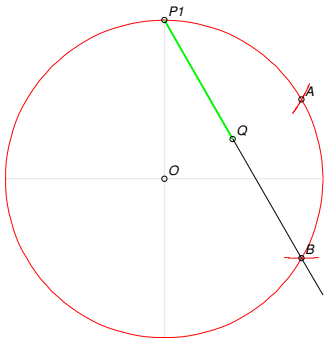

FIGURE 7

4. IMPROVING APPROXIMATE CONSTRUCTIONS

It is also possible to improve approximate constructions for regular polygons like a heptagon where a traditional ruler and compass construction is impossible. In the history of mathematics many ruler and compass constructions have been suggested.

4.1. **Dürer's heptagon.** As an example, let's study the construction of Albrecht Dürer (1525) for a regular heptagon [1]. A line segment corresponding to the side of the heptagon in this construction is obtained by the `GEOM` program of Survo by the following code:

```
-----------------------------------------------------------------------
O=point(0,0)
P1=point(0,1)
C1=circle(O,1)
C2=circle(P1,1,1.8,1.87)
A=cross_cc(C1,C2,1,0.5)
C3=circle(A,1,1.465,1.535)
B=cross_cc(C1,C3,1,-0.5)
P1B=line(P1,B)
Q=midpoint(P1,B)
E=edge(P1,Q)
-----------------------------------------------------------------------
```

In Fig. 7 the line segment P_1Q (in green) is the required side and its length is obtained from the results of `GEOM` in the following way:

```
-----------------------------------------------------------------------
Durer heptagon:
Loading the side length from the results of GEOM:
FILE LOAD -_Edges / VARS=E
   0.8660254037844386

              0.8660254037844386  (side length)
   2*sin(pi/7)=0.8677674782351162  (length in regular heptagon)

An algebraic expression is obtained by INTREL command:
INTREL 0.8660254037844386
X=0.8660254037844386 is a root of 4*X^2-3=0
```

Thus the exact side length in this approximation is $a = \sqrt{3}/2$ corresponding to a central angle $b = 2\arcsin(\sqrt{3}/4) \approx 0.89566$ radians or 51.3178 degrees and the relative error when compared to the true value 360/7 degrees is about -0.002.

It should, however, to be observed that when there are 6 central angles of size b, the remaining central angle is of size $2\pi - 6b \approx 0.90919$ radians or 52.0931 degrees and having a much larger relative error of about 0.013. When the splitting technique is used for improvement there will be still more variation in central angles and sides of the polygon.

Thus we must have a measure for the accuracy of the entire construction. Such an overall measure can be based on the ratio of the areas A = area of the approximate polygon and A_{opt} = area of the regular polygon. More precisely, since these areas are quadratic entities, (and in the spirit of this study) it is reasonable to use the square root of the proportional error

$$M = \sqrt{|1 - A/A_{opt}|} \tag{5}$$

as an overall measure for the accuracy of an approximate construction.

The sucros `NGON-IN` and `NGON-OUT` are using a special for calculating and gathering all pertinent information about iteration rounds and saves that in a text file `_POLSECT.TXT`. In Dürer's construction the results of first 24 iteration rounds are

```
-------------------------------------------------------------------
LOADP _POLSECT.TXT
Round Area              M                 Growth of area    Vertex
    0 2.736379433014045 0.003352522277692
    1 2.736397396679147 0.002162110715972 0.000017963665103  1
    2 2.736401875586612 0.001742968891513 0.000004478907465  2
    3 2.736406354494076 0.001183705354244 0.000004478907465  7
    4 2.736407472714228 0.000996249685177 0.000001118220151  6
    5 2.736408590934378 0.000764112887668 0.000001118220151  3
    6 2.736408870678005 0.000694001695867 0.000000279743627  7
    7 2.736409150421633 0.000615961202214 0.000000279743627  2
    8 2.736409429787981 0.000526607942402 0.000000279366348  4
    9 2.736409709343069 0.000418514843475 0.000000279555088  3
   10 2.736409779255418 0.000386789019453 0.000000069912349  6
   11 2.736409936478628 0.000303561749636 0.000000157223210  5
   12 2.736410006390976 0.000258071322145 0.000000069912349  2
   13 2.736410045706728 0.000228545769817 0.000000039315752  6
   14 2.736410099235626 0.000180752444562 0.000000053528898  7
   15 2.736410116710767 0.000162127390412 0.000000017475141  3
   16 2.736410134182961 0.000141068112116 0.000000017472194  4
   17 2.736410147563212 0.000122517356398 0.000000013380251  6
   18 2.736410155822511 0.000109508928129 0.000000008259299  5
   19 2.736410160191665 0.000101958481157 0.000000004369154  2
   20 2.736410170021017 0.000082483155236 0.000000009829352  3
   21 2.736410173366326 0.000074705778666 0.000000003345309  7
   22 2.736410178895803 0.000059667819228 0.000000005529477  6
   23 2.736410181353297 0.000051596283580 0.000000002457494  2
   24 2.736410183418394 0.000043674969186 0.000000002065097  1
```

This table shows how the area A grows towards $A_{opt} = \frac{7}{2}\sin(2\pi/7) \approx 2.73641$. The original overall accuracy is about 0.003 and it improves to 0.001 in 3 first splits. The last column indicates the index of the vertex that is moved in the current round. The preceding column shows the growth of the area and naturally the three first splits are most influential. Since the 'last' side (from 7 to 1) is larger than the others, at first vertex 1 is moved towards vertex 7. Then the side from 1 to 2 becomes larger than the side from 2 to 3 implying vertex 2 to move towards 1. At the round 3 the sides from 1 to 7 and from 6 to 7 are equally imbalanced as the sides in the previous round. The construction then proceeds symmetrically to vertices 6,3 then to 7,2 until the 'unpaired' side 4,5 breaks the symmetry of improvements.

On all n values the vertex 1 is naturally moved at first. If n is even, this vertex is moved never after this first round and the vertex $n/2 + 1$ (opposite to vertex 1) keeps its original position in all rounds. If $n = 4$, the optimum result, a square, is obtained on the round 3 irrespective of a biased original side length. In all other cases the regularity is not achieved in finite steps when the original side length deviates from the true one.

4.2. **"Stonehenge construction".** Nominally the best approximate heptagon construction I have found in the literature is presented in paper [2] related to a study of the famous prehistoric monument Stonehenge in England. The authors Anthony Johnson and Alberto Pimpinelli tell their main intention on page 3 as follows:

"In this paper, we will focus on the Aubrey Hole circuit. We will argue that it was laid down with the specific purpose of drawing a 56-sided polygon, and that a geometrical construction based on the circle and square, readily doable with pegs and ropes, allows one to trace the polygon to an extremely high accuracy. As a matter of fact, we will show that the method discussed here provides the best known approximation to such a polygon, as well as an exceedingly accurate regular heptagon."

The construction of the side of the heptagon by this approach can be described by the following `GEOM` code

```
---------------------------------------------------------------
O=point(0,0)
C1=circle(O,1)
A=point(0,1)
OA=line(O,A)
A3=cross_cl(C1,OA,0,-1)
LY0=perpendicular(OA,O)
A4=cross_cl(C1,LY0,1,0)
A4A=line(A4,A)
M=midpoint(A4,A)
OM=line(O,M)
F=cross_cl(C1,OM,0.7,0.7)
G=cross_cl(C1,OM,-0.7,0.7)
FF'=perpendicular(ly0,F)
B=cross(FF',A4A)
F'=cross_cl(C1,FF',0.7,-0.7)
GF'=line(G,F')
C=cross(GF',OA)
```

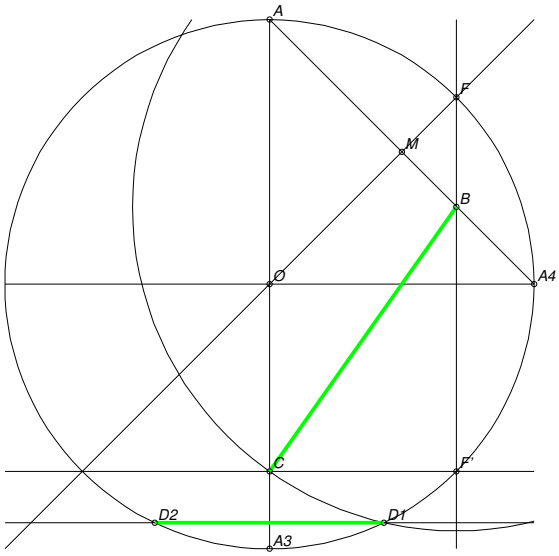

FIGURE 8

```
BC=edge(B,C)
C2=circle(B,BC)
D1=cross_cc(C1,C2,0.3,-1)
D1D2=perpendicular(OA,D1)
D2=cross_cl(C1,D1D2)
E=edge(D1,D2)
```

---

illustrated in Fig. 8. There the line segment D_1D_2 is the side length given by this construction and according to [2] (p.6) its length is

$$\frac{1}{4}(\sqrt{2}-1+\sqrt{15-4\sqrt{2}}) \approx 0.8677173844.$$

The numerical value obtained from the `GEOM` construction confirms it:

```
-----------------------------------------------------------------------
"Stonehenge" heptagon:
Loading the side length from the results of GEOM:
FILE LOAD -_Edges / VARS=Edge,E
 BC                 1.2247448713915889
 E                  0.8677173843843491

               0.8677173843843491  (side length)
   2*sin(pi/7)=0.8677674782351162  (length in regular heptagon)
-----------------------------------------------------------------------
```

The `MAT #POLSECT` operation of Survo gives in this case following results for 12 first splits:

```
-----------------------------------------------------------------------
LOADP _POLSECT.TXT
Round Area              M                 Improvement       Vertex
```

```
 0 2.736410163258900 0.000096304902693
 1 2.736410178063983 0.000062162940385 0.000000014805083  1
 2 2.736410181764966 0.000050117214140 0.000000003700984  7
 3 2.736410185465950 0.000034047602855 0.000000003700984  2
 4 2.736410186391160 0.000028655339791 0.000000000925210  6
 5 2.736410187316370 0.000021977661975 0.000000000925210  3
 6 2.736410187547677 0.000019962170283 0.000000000231307  2
 7 2.736410187778984 0.000017718884246 0.000000000231307  7
 8 2.736410188010282 0.000015147039516 0.000000000231298  5
 9 2.736410188241584 0.000012037652924 0.000000000231302  6
10 2.736410188299411 0.000011125326100 0.000000000057826  7
11 2.736410188357237 0.000010131178844 0.000000000057826  3
```

The initial accuracy of the construction is about 0.001 and the three first splits improves it to 0.0005.

For me, it is hard to believe that this elegant construction could have been invented in those early times. In [2] no measurements are given about the accuracy of the 56-sided polygon. I think that it is much simpler to assume that after the large circle had been 'drawn' by pegs and ropes, the vertices of heptagon were originally placed by some crude guess of the side length, say a_1. Then the difference of the last side from the others corresponding to the central angle $2\pi - 6 \cdot 2\arcsin(a_1/2)$ was divided into 7 equal parts and the side length was corrected according this partition. This practical procedure can be described in a Survo edit field as follows:

```
---------------------------------------------------------------------
Johnson and Pimpinelli (p.1):

"Averaging just over 1 m in width and 1 m deep the holes were found to
have been set on an accurate circle just over 87 m in diameter ---
running just inside the now much weathered and almost invisible 5,000
year-old chalk bank."

Assume that the accuracy of measurements is 1 cm.
radius 87/2=43.5 m
R=4350 cm

Correct side length would be a_opt=round(2*R*sin(pi/7)) a_opt=3775 cm

Let the original rope length be a1=3650 cm
Placing the pegs using this rope length, the length of
the last side d1 is obtained through these calculations:

Central angle  b1=2*arcsin(a1/2/R) b1=0.8658775182941682
Last angle c1=2*pi-6*b1 c1=1.0879201974145776
Its side length d1=round(2*R*sin(c1/2)) d1=4502
The above calculations were not needed in practice.
d1=4502 was obtained by measuring length between
the last two pegs by another rope.
```

```
The difference between rope lengths could then be observed as
error1=d1-a1  error1=852
Then a better side length a2 is obtained by extending the original
rope length by error1/7 by selecting a new rope length
a2=round(a1+error1/7)  a2=3772 cm

Then the same procedure is repeated with a rope of length a2:
Central angle  b2=2*arcsin(a2/2/R) b2=0.8968864596125069
Last angle c2=2*pi-6*b2 c2=0.901866549504545
Its side length d2=round(2*R*sin(c2/2)) d2=3792

a3=round(a2+(d2-a2)/7)  a3=3775 cm  (same as the optimal length)

The M values for a1,a2,a3 are
a1   0.0560
a2   0.0012
a3   0.0001

obtained, for example, for a3 by the sucro command
/APPR_NGON 7,2*arcsin(3775/4350/2),0
giving among other things M=0.000093453793399
```

By three rounds of placing the pegs on the circumference of the circle, the best possible solution was found within the limits of accuracy in measurements. According to M value, the construction was not as good as that of Johnson and Pimpinelli in theory but simpler and accurate enough in practice. It was also achieved without any knowledge of plane geometry although it could have been performed as a traditional ruler and compass construction.

If the above practical construction is iterated in a unit circle by making calculations in double precision, it converges to the to the side of a regular heptagon $2\sin(\pi/7) \approx 0.8677674782351162$. When starting from any reasonable approximate side length, the above approximate value is obtained already after 5 or 6 iterations.

The current version of this paper can be downloaded from
`http://www.survo.fi/papers/Polygons2013.pdf`

References

[1] Albrecht Dürer's construction for a regular heptagon, (1525)
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Department of Mathematics and Statistics, University of Helsinki
E-mail address: seppo.mustonen@survo.fi